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THE
FIRST SIX
AND THE
ELEVENTH AND TWELFTH BOOKS
OF
EUCLID'S ELEMENTS;
WITH
THE ELEMENTS OF PLANE TRIGONOMETRY,
AND
AN APPENDIX IN FOUR BOOKS.

WITH NOTES AND ILLUSTRATIONS.

BY JAMES THOMSON, LL.D.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF GLASGOW.

THIRD EDITION.

LONDON:
SIMMS AND M'INTYRE,
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AND DONEGALL-STREET, BELFAST.
1845.

BIBLIOTHECA · BODLEIANA

EASY EXERCISES

ON THE

FIRST PART OF THIS WORK.

EXERCISES ON THE FIRST BOOK OF EUCLID.

1. Prove Euc. I. 9, I. 11, and I. 12, without using I. 8.

2. If there be two unequal parallel straight lines, the two straight lines, joining their extremities which are towards the same parts, will meet if produced through the extremities of the shorter parallel.

3. The three straight lines drawn from a point within a triangle to the angles, are together less than the perimeter, but greater than its half.

4. The diagonals of a parallelogram divide it into four equal parts.

5. If a quadrilateral be bisected by both diagonals, it is a parallelogram.

6. Prove Euc. I. 32, by the construction of Euc. I. 16.

7. Given the perpendicular of an equilateral triangle, to construct it.

8. Given the diagonal of a square, to construct it.

9. Prove Euc. I. 17, without producing any side.

10. Prove the first and second parts of Euc. I. 28, independently of I. 27, and of one another: and from each, when proved, derive I. 27, and the remaining part of I. 28.

11. Given the sum of the side and diagonal of a square; to construct it.

12. Given the difference of the side and diagonal of a square; to construct it.

13. Given the sum of the diagonal and two sides of a square; to construct it.

14. Given the sum of the side and perpendicular of an equilateral triangle; to construct it.

15. Given the difference of the side and perpendicular of an equilateral triangle; to construct it.

16. Prove Euc. I. 36, without joining BE, CH, by producing BA, FE, and CD, GH till they meet. When will this fail, and how will the proposition then be proved?

17. If a diagonal of a parallelogram be either equal to a side or less than a side of the same figure, that diagonal is less than the other.

18. If a rectangular floor be 25 feet 6 inches (25·5 feet) long, and 16 feet 9 inches (16·75 feet) broad, what is its area in square yards? *Answ.* 47·4583′.

19. If the length of a rectangular garden be 15 perches, what must be its breadth, so that it may contain an acre? *Answ.* 10⅔ perches.

20. If the side of a square be 89 inches, what is its area? *Answ.* 7921 square inches.

21. Find the side of a square field, containing three acres. *Answ.* 21·9089 perches.

22. Given the base of a parallelogram equal to 14 chains 32 links, and its perpendicular breadth equal to 10 chains 46 links; to find its area.[*] *Answ.* 14a. 3r. 36·6p. nearly.

23. If the base and perpendicular of a triangle be 66 and 32 yards respectively, what is the area? *Answ.* 1056 square yards.

24. Given the diagonal of a four-sided field equal to 9 chains 62 links, and the perpendiculars to it from the angles which it subtends, equal to 4 chains 20 links, and 5 chains 77 links, respectively; to compute the area. *Answ.* 4a. 3r. 7·3p. nearly.

25. Given the legs of a right-angled triangle, equal to 2263 and 2184 feet, respectively; to find the hypotenuse. *Answ.* 3145 feet.

26. Given the hypotenuse of a right-angled triangle equal to 79 chains 13 links, and one of the legs equal to 58 chains 65 links; to find the other leg. *Answ.* 53 chains 12 links.

27. The square described on the diagonal of a square is double of the square itself; and a square is double of the square described on half its diagonal.[†]

28. Given the legs of a right-angled triangle, equal to 136 and 255 inches, respectively; to compute the length of the perpendicular from the right angle to the hypotenuse. *Answ.* 120 inches.

29. Through two given points in two parallels, to draw two straight lines forming a rhombus with the parallels.

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[*] The land-surveyor's chain, invented by Gunter of Gresham College, London, is four perches in length, and is divided into 100 equal links. Such being the length of the chain, the area of a square chain is 16 square perches, and accordingly 10 square chains make an acre. To find, therefore, the area of a parallelogram in acres, multiply the length by the perpendicular breadth, taking the chains and links in each as a single number, and from the product cut off five figures as decimals. Thus, in the present example, the dimensions are 14·32 and 10·46 chains, and the area is 14·32×10·46, or 149·7872 square chains; and by dividing this by 10, we get 14·97872 acres, the same as would be obtained by multiplying 1432 by 1046, and cutting off five figures. The roods are found by multiplying the decimal by 4, the roods in an acre, and cutting off five figures; and the perches by multiplying those figures by 40, the square perches in a rood, and cutting off five figures.

[†] Hence, if the side of a square be given, the diagonal will be found by multiplying the square root of 2 by the side: and if the diagonal be given, the side will be found by multiplying the square root of 2 by half the diagonal. These conclusions are easily illustrated by means of algebra.

PREFACE.

As the price of the former editions of this work was found to be an obstacle with many who would have wished to employ it as a text-book, the present edition, though with several improvements and additions, is published in a different and a cheaper form. It is done up either in ONE VOLUME, or in TWO PARTS; so that, at the pleasure of the teacher or student, it may be used either as a whole, or one part may be first studied, and after that the other, if the learner wish to obtain a more profound knowledge of elementary geometry. The First Part contains the First Six Books of Euclid, with the Elements of Plane Trigonometry: and the Second comprehends the Eleventh and Twelfth Books of Euclid, with an Appendix in Four Books. This Appendix, besides containing a number of miscellaneous propositions, several of which are curious and valuable, treats of the Tangencies, of Loci, of Porisms, of Isoperimetrical Figures, and of the Quadrature of the Circle; and by means of some of the notes, the student will be made acquainted, to a certain extent, with the nature and character of the ancient Geometrical Analysis. These subjects are necessarily treated of with brevity. What is here given, however, will perhaps be nearly sufficient for the mathematical student of the present day, whose time and attention, after he has acquired a moderate acquaintance with the ancient geometry, will be more profitably devoted to the study of modern science. In addition to what has been already mentioned, the Second Part contains Notes and Illustrations on the preceding part of the work, which will be useful not only to the teacher, but also to the student who wishes to have an extensive knowledge of elementary geometry. In this part also, there are given many exercises, partly new and partly selected, which it may be useful for the student to perform from time to time, when he finds that he has got principles and practice to enable him to do so.

In this edition, a number of simple and easy exercises—many which are of a practical nature—are prefixed to the First Part, which ought to be performed by every student who wishes to obtain even a moderate knowledge of geometry, and the elementary parts of plane trigonometry: and to the Second Part a short article is subjoined, giving a view of the leading facts regarding the History of Elementary Geometry and Trigonometry.

With regard to the Eight Books of Euclid contained in the work, it is proper to state, that Dr. Simson's valuable edition has been generally followed. In the definitions, however, several considerable changes, and, it is hoped, improvements, have been made; and, in various instances throughout the work, repetitions have been done away, that seemed to be unnecessary, and which must often have been embarrassing to the learner. Besides the more lengthened Notes at the end of the work, various foot-notes have been given, which will assist the student, and increase his knowledge. The demonstrations, also, of several propositions have been shortened and simplified, particularly those of the 5th and 35th propositions of the First Book of Euclid; of the 13th and others in the Second Book; of the 14th, 15th, 21st, 31st and 32d of the Third Book; and of several in the remaining Books.

The short article on Trigonometry, at the end of the First Part, contains the best methods at present known for the resolution of triangles; and it gives the method of determining the heights and distances of objects in some of the simplest and most useful cases. Throughout the work, indeed, care has been taken to give the matter, directly or indirectly, when opportunities occurred, a practical character, without interfering, however, with the strictness of reasoning, which is so well calculated to train and strengthen the mind of the student, and thus to promote one great object of scientific education.

To the Fifth Book a Supplement is annexed, in which the principles of proportion are established in a brief and simple way, and which, for saving the time of the student, may be used with advantage, instead of the method employed by Euclid, which is prolix, and which is generally found by beginners to be somewhat difficult.

Of the improvements and additions, some are new, and some are derived from the writings of others. Much originality cannot be expected in a work on a subject, which, for more than twenty centuries, has occupied the attention, and been cultivated by the talents and labour of a great number of eminent men. Besides this, the great object in view in the present publication has been utility; and, accordingly, the writings of others have been carefully consulted, and whatever appeared to be valuable and appropriate, has been freely but not servilely used.

Suggestions have been given by several individuals, which have contributed to the improvement of the work. From Mr. Michael Lawler of Dublin, in particular, several judicious remarks have been received, which have been employed with advantage.

Glasgow College, Feb. 14, 1845.

EXERCISES ON THE SECOND BOOK OF EUCLID.

1. Given the hypotenuse and one leg of a right-angled triangle equal to 2045 and 1924; to find the remaining leg without squaring the given numbers. *Answ.* 693.

2. If a side of a parallelogram be equal to one of the diagonals, the squares of the two sides which intersect that side are together equal to the difference of the squares of the diagonals.

3. If the sides of a triangle be 2535, 2730, and 2925, respectively, what are the lengths of the segments into which they are severally divided by the perpendiculars from the opposite angles.* *Answ.* 1485 and 1050, 1755, and 975, and 1638 and 1287.

4. If the sides of a triangle be 8500, 4080, and 5780, what will be the segments of the several sides made by perpendiculars from the opposite angles. *Answ.* 5236 and 3264, 6800 and —2720, and 7700 and —1920; the mark — denoting that the segment to which it is prefixed is not in the base, but in its continuation.

5. If the base of a triangle be 70, and the other sides 45 and 35; what is the length of the straight line drawn from the vertical angle to the point of bisection of the base? *Answ.* 20.

6. If two adjacent sides and one of the diagonals of a parallelogram be 100, 120, and 150, what is the length of the other diagonal? *Answ.* 162·17, nearly.

7. Given the sides of a triangle equal to 21, 17, and 10 feet, respectively; to compute the area. *Answ.* 84 square feet.

EXERCISES ON THE THIRD BOOK OF EUCLID.

1. In a given line, straight or curved, find a point from which, as centre, if a circle be described, it will pass through two given points. When will this be impossible, and when will there be more than one solution?

2. Through one extremity of a given straight line, to draw a perpendicular to it, without producing it.

3. To describe a circle touching a given arc of a circle, and the lines passing through its extremities, and the centre.

4. How many points determine the position of a straight line, and how many the position and magnitude of a circle?

5. If the arc of a segment of a circle be a fourth of the circumference, what is the magnitude of an angle in the segment?

6. If a chord of a circular arc, 17 inches in length, be divided into two parts of 8 and 9 inches, respectively, by another chord, what is the length of the latter, one of its segments being 4 inches? *Answ.* 22 inches.

7. If the chord of an arc be 336 feet, and the chord of its half 175 feet, what is the diameter of the circle? *Answ.* 625 feet.

8. If from a point without a circle, two straight lines be drawn

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* This should be solved both by means of the thirteenth proposition and of the fourth corollary to the fifth. The next should be done by means of the twelfth and thirteenth propositions, and the same corollary.

cutting it, and if the parts of one of them, without the circle and within it, be respectively 8 inches and 4 inches, while the external part of the other is 6 inches; what is the length of the part of the latter within the circle? *Answ.* 10 inches.

9. Suppose that, with a view to find the diameter of a circular pond, a point is taken outside, such that tangents from it to the circumference, form with each other an angle of an equilateral triangle, and that the length of each tangent is found to be 12 perches, what is the diameter? *Answ.* 13·8564 ($=8\sqrt{3}$) perches.

10. In a given straight line, to find a point, from which, as centre, if a circle be described with a given radius, it will touch another straight line given in position. When will there be two solutions? When only one? and when will the problem be impossible?

11. From a given point as centre, to describe two circles, each of them touching a given circle. What different cases of this problem may there be?

12. Through a given point, either within or not within a given circle, to draw a straight line cutting the circle, so that the part of it within the circle may be equal to a given straight line. What are the limits of the magnitude of the given line?

EXERCISES ON THE FOURTH BOOK OF EUCLID.

1. If a circle be inscribed in a triangle, and if another triangle be formed by joining the points of contact, each angle of the latter triangle is equal to half the sum of the two angles of the original triangle at the extremities of the side on which its vertex stands.

2. In a given circle, to inscribe a quadrilateral having two of its sides and the diagonal drawn through their point of intersection, respectively equal to given straight lines. When will this be impossible, and when will there be only one solution, and when two?

3. Given two opposite sides of a quadrilateral described in a given circle, and one of the diagonals; to construct the figure. What cases may there be?

4. In an equilateral triangle, to describe three circles, each touching the other two, and touching two sides of the triangle.

5. About a square to describe an equilateral triangle.

6. In a square to inscribe an equilateral triangle, having one of its angular points at one of the angles of the square.

7. If two remote sides of a regular pentagon be produced till they meet, the angle which they form is two fifths of a right angle.

8. In a given square, to describe two equal semicircles touching each other, and having their diameters coinciding with two adjacent sides, and each of them touching also one of the remaining sides of the square.

9. If the sides of a triangle be 13, 14, and 15 yards, respectively, what is the radius of its inscribed circle? And what are the radii of the circles, each touching one of the sides and the other two produced? *Answ.* 4, 10½, 12, and 14 yards, respectively.

EXERCISES ON THE FIFTH BOOK OF EUCLID.

1. If $A : B :: C : D$, prove that $mA+B : B :: mC+D : D$.

2. On the same hypothesis, prove that $mA+nB : pA+qB :: mC+nD : pC+qD$.

3. On the same hypothesis, still, prove that $A^2+B^2 : A^2-B^2 :: C^2+D^2 : C^2-D^2$.

4. Prove also that $A^2+AB+B^2 : A^2-AB+B^2 :: C^2+CD+D^2 : C^2-CD+D^2$.

5. Find a third proportional to 4 and 6. *Answ.* 9.

6. Find a fourth proportional to 12, 16, and 39. *Answ.* 52.

7. Find a mean proportional between 24 and 54. *Answ.* 36.

EXERCISES ON THE SIXTH BOOK OF EUCLID.

1. If the sides of a triangle be 532, 684, and 912, respectively, what are the parts into which each of them is divided by the straight line bisecting the opposite angle? *Answ.* 228 and 304; 252 and 432; and 399 and 513.

2. Given the legs of a right-angled triangle equal to 65 and 156 inches, respectively; to compute the segments into which the hypotenuse is divided by the perpendicular to it from the right angle, and to compute also that perpendicular. *Answ.* The segments are 25 and 144, and the perpendicular 60.

3. If the base of a triangle be 123, and the other sides 125 and 80, respectively, what is the length of the line bisecting the vertical angle? *Answ.* 80.

4. If two homologous sides of two similar fields be 15 and 25 perches respectively, and if the content of the greater be 5 a. 1 r. 35 p. what is the content of the less? *Answ.* 1 a. 3 r. 35 p.

5. If the homologous sides of two similar fields be in the ratio of 3 to 4, and if the one contain two acres more than the other, what are their respective contents? *Answ.* 2 a. 2 r. $11\frac{3}{7}$ p; and 4 a. 2 r. $11\frac{3}{7}$ p.

6. If the joint content of two similar fields be 7 acres, what are their separate contents, their homologous sides being as 5 to 7? *Answ.* 2 a. 1 r. $18\frac{14}{37}$ p. and 4 a. 2 r. $21\frac{23}{37}$ p.

7. If the base of a triangle be 18, and the other sides 25 and 16, respectively, what is the length of the external segment of the base, made by a straight line bisecting the exterior angle at the vertex? *Answ.* 32.

8. From the same data find the length of the bisecting line. *Answ.* 28·28427 ($=20\sqrt{2}$).

9. If the sides of a triangle be 13, 14, and 15 inches, respectively, what is the radius of the circumscribed circle? *Answ.* $8\frac{1}{8}$.

10. Through a given point to draw a straight line, forming with two sides of a given triangle, or with one or both of them produced, a triangle similar to the given one. Show that in general this may be done in six different ways, and point out in what circumstances there will be fewer solutions.

EXERCISES IN TRIGONOMETRY.

1. Given the sides of an isosceles triangle, to compute its angles in the simplest manner.

2. Given the angles and any side of an isosceles triangle, to find the remaining sides in the easiest manner.

3. Given the angles of a triangle, and the straight line drawn from the vertical angle to the point of bisection of the base; to compute the sides.

4. Prove that the sine of twice a circular arc, is less than twice its sine; but that the tangent of twice an arc is greater than twice the tangent of the same arc; the simple arc in each instance being less than 45°. What varieties will there be in each case, if the arc exceed 45°?

5. Prove the third proposition of the sixth book of Euclid by means of trigonometry.

6. Given two angles of a trapezium, and the three sides which form them; to show the method of computing the remaining side.

7. Compute the angle which a straight line drawn from one angle of a square, to bisect one of the remote sides, makes with that side. *Answ.* 63° 26′, nearly.

8. If the natural sine of an angle be 0·50528, what is its cosine? *Answ.* 0·86295.

9. If the natural cosine of an angle be 0·86603, what is its cotangent? *Answ.* 1·73206.

10. If the natural secant of an angle be 1·4142136, what are its sine and tangent? *Answ.* 0·7071068 and 1.

11. If the natural tangent of an angle be 0·51872, what is its sine? *Answ.* 0·46046.

THE
ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.[*]

1. A *point* is that which has position, but not magnitude.[†]

2. A *line* is length without breadth.

Corollary. The extremities of a line are points; and an intersection of one line with another is also a point.[‡]

3. If two lines be such that they cannot coincide in any two points, without coinciding altogether, each of them is called a *straight* line.[‖]

Cor. Hence two straight lines cannot enclose a space. Neither can two straight lines have a common segment; that is, they cannot coincide in part without coinciding altogether.[§]

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[*] *Definitions* explain the precise sense in which terms are to be understood, distinguishing the ideas expressed by those terms from the ideas expressed by any others.

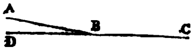
[†] In geometrical figures, or, as they are also called *diagrams*, we are obliged, instead of mathematical points and lines, to employ physical ones; that is, dots, instead of mathematical points, and lines of some perceptible breadth, instead of mathematical ones; as the finest line that we can make has breadth, and the finest point both length and breadth. Our reasoning, however, is not vitiated on this account, as it is conducted on the supposition that the point has no magnitude, and the line no breadth, and nothing in it depends on the magnitude of the point or the breadth of the line in the diagram.

[‡] A *corollary* is an inference which arises easily and immediately from some other principle, not requiring any lengthened process of reasoning to establish its truth.

The truth of this corollary is manifest, since, from the definition of a line, it follows, that the extremities and the intersections of lines have position, but not magnitude, and therefore (definition 1) are points. The corollaries to the fourth and sixth definitions may be illustrated in a similar manner.

[‖] This will be illustrated by applying the edges of two straight rulers to one another, and turning them in different ways. It will also be rendered clearer by a negative illustration, by means of two wires, or other substances, bent similarly; as, though these may coincide, entirely when turned in one direction, they will coincide only in some points when placed otherwise.

[§] Thus, if ABC, DBC could both be straight lines, they would have the segment or part, BC common. But this is impossible; for if they coincide in any two points, as B and C, the parts AB and DB must also coincide.



4. A *surface*, or *superficies*, is that which has only length and breadth.

Cor. The extremities of surfaces are lines; and an intersection of one surface with another is also a line.

5. A *plane* surface, or, as it is generally called, a *plane*, is that in which any two points being taken, the straight line between them lies wholly in that surface.*

Cor. Hence two plane surfaces cannot enclose a space. Neither can two plane surfaces have a common segment.

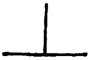
6. A *body*, or *solid*, is that which has length, breadth, and thickness.

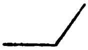
Cor. The extremities of a body are surfaces.

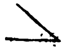
7. A *rectilineal angle* is the mutual inclination of two straight lines, which meet one another.† The point in which the straight lines meet is called the *vertex* of the angle.

8. When one straight line standing on another makes the adjacent angles equal, each of the angles is called a *right angle*; and the straight line which stands on the other is called a *perpendicular* to it, and is said to be *at right angles* to it.

9. An *obtuse* angle is that which is greater than a right angle.

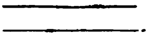






10. An *acute* angle is that which is less than a right angle.

11. *Parallel* straight lines are those which are in the same plane, and which, being produced ever so far both ways, do not meet.



12. A *figure* is that which is enclosed by one or more boundaries.

13. *Rectilineal* or *rectilinear* figures are those which are contained by straight lines:

14. *Triangles*, by three straight lines:

15. *Quadrilateral* figures, by four straight lines:

16. *Polygons*, by more than four straight lines.‡

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* On the contrary, if two points be taken on the surface of a ball, the straight line between them will lie within the ball, and not on its surface. The surface of a ball, therefore, is not a plane surface.

† Or a *rectilineal angle* is the degree of opening, or divergence of two straight lines which meet one another. A clear idea of the nature of an angle is obtained by gradually opening a carpenter's rule, or a pair of compasses; as the angle made by the parts of the rule or the legs of the compasses, will become greater as the opening widens. It is evident that the magnitude of the angle does not depend on the length of the lines which form it, but merely on their relative positions.

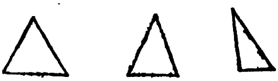
An angle is best named by a single letter placed at its vertex, unless there be more angles than one at the same point. In this case, the angle is generally expressed by three letters, the middle one of which is placed at the vertex, and the others at some other points of the lines containing it. Thus, in the first figure for the fifth proposition of this book, the angle contained by AB and AC is called the angle A, while that which is contained by AB and BC is called the angle ABC.

‡ An *equilateral* figure is that which has equal sides, and an *equiangular* one that which has equal angles. A polygon which is equilateral and equiangular is called a *regular* polygon. Polygons, especially when they are regular, are often distinguished by particular names, derived from the Greek language, denoting the number of their angles, and consequently of their sides. Thus, a polygon of five sides is called a *pentagon;* of six, a *hexagon;* of seven, a *heptagon;* of eight, an *octagon;* of nine, an *enneagon;* of ten, a *decagon;* of twelve, a *dodecagon;* and of fifteen, a *pentedecagon.*

17. Of three-sided figures,* an *equilateral* triangle is that which has its three sides equal:

18. An *isosceles* triangle is that which has two sides equal:

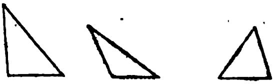
19. A *scalene* triangle is that which has all its sides unequal:



20. A *right-angled* triangle is that which has a right angle:

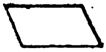
21. An *obtuse-angled* triangle is that which has an obtuse angle:

22. An *acute-angled* triangle is that which has three acute angles.



23. In a figure of four or more sides, a straight line drawn through two remote angles of it, is called a *diagonal.*‡

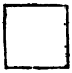
24. Of four-sided figures, a *parallelogram* is that which has its opposite sides parallel.



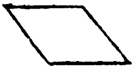
25. Any other four-sided figure is called a *trapezium.*

26. A parallelogram which has a right angle is called a *rectangle.*†

27. A rectangle which has two adjacent sides equal, is called a *square.*



28. A parallelogram which has two adjacent sides equal, but its angles not right angles is called a *rhombus.*§



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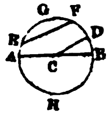
[*] From this and the five following definitions, it appears that triangles are divided into three kinds from the relations of their sides, and into three others from those of their angles. When the three sides are equal, the triangle is equilateral; when two are equal, it is isosceles; and when they are all unequal, it is scalene. Again, a triangle may have one right angle, one obtuse angle, or neither one nor other; and hence the distinction into right-angled, obtuse-angled, and acute-angled. It will appear from the 17th or 32d proposition of this book, that a triangle can have only one right or one obtuse angle; and, from the 32d, that it may have three acute angles. It may be remarked that the term *scalene* is seldom used; and that obtuse-angled and acute-angled triangles are often called *oblique-angled* triangles in contradistinction to right-angled ones.

[†] In Simson's, and most other editions of Euclid, the diagonals of parallelograms are called their *diameters.* It is better, however, to confine the term diameter to the lines which are so called in the circle and other curves. It may be remarked, that in this definition, as in many other instances, the term *angle* is used to denote what in strictness is the *vertex* of the angle.

[‡] A rectangle which is not a square, is sometimes called an *oblong.*

[§] The following are the definitions of the square and rhombus which are given in Simson's edition:—

29. A *rhomboid* is a parallelogram which has not its adjacent sides equal, nor its angles right angles.



30. A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.*

31. That point is called the *centre* of the circle.

32. Any straight line drawn from the centre of a circle to the circumference is called a *radius* of the circle.

33. An *arc* of a circle is any part of the circumference.

34. A straight line drawn from one point in the circumference to another, is called the *chord* of either of the arcs into which it divides the circumference.

35. A *diameter* of a circle is a chord which passes through the centre.

36. A *segment* of a circle is the figure contained by an arc and its chord. The chord is sometimes called the *base* of the segment.

37. A *semicircle* is a segment whose chord is a diameter.†

38. Two arcs which are together equal to the arc of a semicircle are called *supplements* of one another, or are said to be *supplementary*. So also are two angles which are together equal to two right angles.

"A *square* is a four-sided figure which has all its sides equal and all its angles right angles.

"A *rhombus* is a four-sided figure which has all its sides equal; but its angles are not right angles."

The latter of these is a correct definition of the figure; but it does not point the rhombus out as being a species of parallelogram. The former, though it contains nothing false, errs in being redundant, and in ascribing properties to the square, the possibility of its having which requires to be proved. Thus, as will appear hereafter, it can be proved, that if a quadrilateral have all its sides equal, and have one right angle, it will have all its angles right angles: and till the 32d proposition and its corollaries are established, we are no more entitled to conclude that a quadrilateral can have four right angles than that it can have four obtuse or four acute ones. The definition above quoted, however, has the advantage of embodying, in very simple and concise terms, the principal properties of the square: and some may still prefer using it, especially after having demonstrated the 46th proposition.

* According to this definition, which is remarkable for its perspicuity and precision, the *circle* is the space inclosed, while the *circumference* is the line that bounds it. The *circumference*, however, is frequently called the circle.

† The definitions of the radius, arc, and chord are here given on account of the constant use of the terms in mathematics. The terms arc (for which some writers rather improperly use arch) and chord receive their names from the bow (in Latin *arcus*), and its cord or string. The diameter being merely a particular chord, its definition is placed after that of the chord, and made to depend on it. In like manner the definition of a segment of a circle is placed before that of its species, the semicircle. The following, which is Euclid's definition of the semicircle, will perhaps be preferred by some:—

"A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter."

It may be proved that a diameter divides a circle into two equal parts, by inverting one of them and applying it to the other, so that the centres may coincide; as the figures will coincide altogether, the radii being all equal. It is from this that the semicircle gets its name. It is evident that a circle might be divided equally in numberless other ways; but it is only the parts into which it is divided by a diameter, that are called semicircles.

POSTULATES.*

1. Let it be granted, that a straight line may be drawn from any one point, to any other point:†

2. That a terminated straight line may be produced to any length in a straight line:

3. That a circle may be described from any centre, at any distance from that centre.‡

AXIOMS.

1. Things which are equal to the same, or to equals, are equal to one another.

2. If equals or the same be added to equals, the wholes are equal.

3. If equals or the same be taken from equals, the remainders are equal.

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* The *propositions* in mathematics—that is, the subjects proposed to the mind for consideration—are either problems or theorems. In a *problem* something is required to be performed, such as the drawing of a line, or the construction of a figure; and whatever points, lines, angles, or other magnitudes, are given for effecting the object in view, are called the *data* of the problem. A *theorem* is a truth proposed to be demonstrated: and whatever is assumed or admitted as true, and from which the proof is to be derived, is termed the *hypothesis*.

A *postulate* is a problem so simple and easy in its nature, that it is unnecessary to point out the method of performing it; or in strictness, it is the demand of the author that the reader may admit the method of performing it to be known. The method of solving all other problems that occur in the work, is pointed out. An *axiom* is a proposition, the truth of which the human mind is so constituted as to admit, as soon as the meaning of the terms in which it is expressed, is understood. It is plain from this that postulates bear the same relation to other problems, as axioms do to theorems.

Some writers tacitly employ the postulates and axioms without giving them in a separate form. It seems better, however, to let the student know, at the outset, what propositions he is to take for granted without proof. In this way he is not stopped in his progress to consider whether the point tacitly admitted be established in some previous proposition; and though it may be said that axioms are less evident in a general form, than in particular cases, the collection of them into a separate list will not hinder the student from considering in any particular instance, whether an axiom is legitimately applicable or not.

From what has now been said, it will appear that a corollary, as already explained, is likewise a proposition. It may also be remarked that a proposition which is preparatory to one or more others, and which is of no other use, is called a *lemma*. Such a proposition is thrown into a separate form for the sake of simplicity and distinctness.

† To *join* two points is an abbreviated expression, meaning the same as to draw a straight line from one of them to the other.

‡ These postulates require, in substance, that the simplest cases of drawing straight lines, and describing circles be admitted to be known; and they imply the use of the rule and compasses, or something equivalent. A circle may also be described by means of a cord or other line of invariable length, fixed at one extremity, or by employing another circle, already drawn, as a pattern. The latter mode, however, does not agree with the third postulate, as it does not enable us to describe a circle from a given centre, and at a given distance from that centre.

4. If equals or the same be added to unequals, the wholes are unequal.[*]

5. If equals or the same be taken from unequals, the remainders are unequal.

6. Things which are doubles of the same, or of equals, are equal to one another.[†]

7. Things which are halves of the same, or of equals, are equal to one another.

8. Magnitudes which exactly coincide with one another, are equal.

9. The whole is greater than its part.[‡]

10. The whole is equal to all its parts taken together.[§]

11. All right angles are equal to one another.[‖]

12. If a straight line meet two other straight lines which are in the same plane, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines shall at length meet upon that side, if they be continually produced.[¶]

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[*] In this axiom and the following, it is evident that the result obtained by adding to the greater, or taking from it, is greater than that which is obtained by adding to the less, or taking from it.

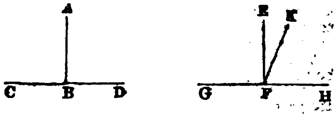
[†] This axiom might be derived from axiom 2; and the next from this one by an indirect demonstration. They are generalized in the first and second axioms of the fifth book.

[‡] This is implied in the significations of the terms whole and part. In strictness, therefore, it is scarcely to be regarded as an axiom.

[§] Though this axiom is not delivered by Euclid in a distinct form, it is tacitly employed by him in many instances. Like the foregoing, it may be regarded as being implied in the signification of the terms whole and part.

[‖] This axiom does not relate to the right angles made by one line standing on another, such angles being equal by the eighth definition. Instead of being considered as an axiom, this proposition has been demonstrated by some writers in the following manner:

Let AB be perpendicular to CD, and EF to GH, then the angles ABD, EFH are equal. For, let the straight line CD, be applied to GH, so that the point B may fall on F, and if the straight line BA do not fall on FE, let it take the position FK. Then (def. 8) ABD is equal to ABC and EFG to EFH. But (axiom 9) GFK or its equal KFH, is greater than GFE or its equal EFH, which is impossible, since (ax. 9) EFH is greater than KFH: therefore BA cannot have the position FK; and in the same manner it might be shown that it cannot have any other position except FE; and therefore (ax. 8) the angles ABD, EFH are equal.



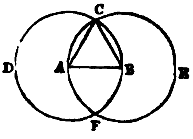
[¶] This will be illustrated in the remarks on the 28th proposition of the first book; and the student may postpone the consideration of it, till he has proved that proposition. He will find it useful also, in other instances, to postpone the serious consideration of definitions, postulates, and axioms, till they come to be employed in the propositions; but he should then make himself perfectly acquainted with their nature.

PROPOSITIONS.

PROPOSITION I. PROBLEM.[*]—To describe an equilateral triangle on a given finite straight line.[†]

Let AB be the given straight line: it is required to describe an equilateral triangle upon it.

From the centre A, at the distance AB, describe (I. postulate 3[‡]) the circle BCD, and from the centre B, at the distance BA, describe (I. post. 3) the circle ACE; and from the point C, in which the circles cut one another, draw (I. post. 1) the straight lines CA, CB to the points A, B: ABC is the equilateral triangle required.



Because the point A is the centre of the circle BCD, AC is equal (I. definition 30) to AB; and because the point B is the centre of the circle ACE, BC is equal (I. def. 30) to BA. But it has been proved that CA is equal to AB; therefore, CA, CB are each of them equal to AB: but things which are equal to the same are equal (I. axiom 1) to one another; therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore (I. def. 17) equilateral, and it is described upon the given straight line AB: which was required to be done.

Scholium.[§] If straight lines be drawn from A and B, to F, the other point in which the circles cut one another, it would be proved in the same manner that AFB is an equilateral triangle. Hence on any straight line two equilateral triangles may be described one on each side of it.*

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[*] The words in which a proposition is expressed, are called its *enunciation*. If the enunciation refer to a particular diagram, it is called a *particular* enunciation:—otherwise, it is a *general* one.

A *demonstration* is a series of arguments which establish the truth of a theorem, or of the solution of a problem. Demonstrations are either *direct* or *indirect*. The *direct* demonstration commences with what has been already admitted, or proved, to be true, and from this deduces a series of other truths, each depending on what precedes, till it finally arrives at the truth to be proved. In the *indirect* or *negative* demonstration, or as it is also called, the *reductio ad absurdum*, a supposition is made which is contrary to the conclusion to be established. On this assumption, a demonstration is founded, which leads to a result contrary to some known truth; thus proving the truth of the proposition, by showing that the supposition of its contrary leads to an absurd conclusion.

The drawing of any lines, or the performing of any other operation that may be necessary in a proposition, is called the *construction*.

In this proposition, the first paragraph is the general enunciation; the second, the particular one; the third, the construction; and the fourth, the demonstration.

[†] That is, to describe an equilateral triangle, which shall have given a straight line as one of its sides. The word *finite* is employed to show that the line is not of unlimited length, but is given in magnitude, as well as position.

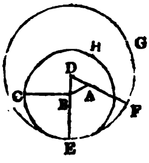
The student should be accustomed to point out the data in problems and the hypotheses in theorems. In this problem, a straight line is given, and it is required to describe on it an equilateral triangle.

[‡] In the references, the Roman numerals denote the book, and the others, when no word is annexed to them, indicate the proposition;—otherwise the latter denote a definition, postulate, or axiom, as specified. Thus, III. 16 means the sixteenth proposition of the third book; and I. ax. 2, the second axiom of the first book. So also *hyp.* denotes *hypothesis*, and *const.* *construction*.

[§] By a *scholium* is meant a note or observation.

PROP. II. PROB.—From a given point to draw a straight line equal to a given straight line.†

Let A be the given point, and BC the given straight line; it is required to draw from A a straight line equal to BC.



From the point A to B draw (I. post. 1) the straight line AB; and upon it describe (I. 1) the equilateral triangle DAB, and produce (I. post. 2) the straight lines DB, DA, to E and F. From the centre B, at the distance BC, describe (I. post. 3) the circle CEH, and from the centre D, at the distance DE, describe (I. post. 3) the circle EFG. AF is equal to BC.‡

Because the point B is the centre of the circle CEH, BC is equal (I. def. 30) to BE; and because D is the centre of the circle EFG, DF is equal (I. def. 30) to DE; and DA, DB, parts of them are equal: therefore the remainder AF is equal (I. ax. 3) to the remainder BE. But it has been shown, that BC is equal to BE; wherefore AF and BC are each of them equal to BE; and things that are equal to the same are equal (I. ax. 1) to one another; therefore the straight line AF is equal to BC. Wherefore from the given point A a straight line AF has been drawn equal to the given straight line BC: which was to be done.

PROP. III. PROB.—From the greater of two given straight lines to cut off a part equal to the less.§

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[*] It will be shown in the 10th proposition of the 3d book, that two circles can cut each other in only two points; and hence there can be only two equilateral triangles on a straight line.

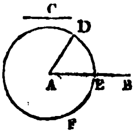
In the practical construction it is sufficient to describe small arcs intersecting each other in C or F.

[†] Here the data are a point and a straight line.

[‡] A straight line may be drawn from the point A, to either extremity of BC, and on either of the lines thus drawn, two equilateral triangles may be constructed. Hence there may be four straight lines drawn, any one of which will be such as is required in the problem. To this there is an exception, when A is at either extremity of BC, or in its continuation, as in either case it will readily appear that only two such lines, can be drawn by the construction here given. It is plain also that if A be in BC, the equilateral triangle is to be described on either of the parts into which BC is divided at that point. In practice, every person will solve this problem by opening the compasses to the distance between B and C; and then, one point being placed at A, the other will mark out the point F on any line drawn from A. This method, however, though greatly preferable in practice, could not in strictness be adopted by Euclid, as it is not derived from the postulates or the first proposition. It is the same in fact as assuming this proposition as an additional postulate. In a science, however, which is strictly demonstrative, the fewer first principles that are assumed the better; and though it may sometimes be convenient, and may save time, to dispense with strict rigour, it is always satisfactory to know that the object in view may be attained without any departure from geometrical accuracy.

[§] Here the data are two straight lines. In practice, what is done in this propo-

Let AB and C be the two given straight lines, of which AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.



From the point A draw (I. 2) the straight line AD equal to C; and from the centre A, at the distance AD, describe (I. post. 3) the circle DEF: AE is the part required.

Because A is the centre of the circle DEF, AE is equal (I. def. 30) to AD; but the straight line C is likewise equal (const.) to AD; whence AE and C are each of them equal to AD; wherefore the straight line AE is equal (I. ax. 1) to C, and from AB, the greater of two straight lines, a part AE has been cut off equal to C the less: which was to be done.

Schol. By drawing (I. 2) from either extremity of C, a straight line equal to AB, the line C might be produced, till it and the part added would be equal to AB. Also, by producing AB through A, to meet the circumference, a line would be obtained equal to C and AB together.

PROP. IV. THEOREM.—**If two triangles have two sides of the one equal to two sides of the other, each to each;* and have also the angles contained by those sides equal to one another: (1) they have likewise their bases, or *third sides,*† equal; (2) the two triangles are equal;‡ and (3) their other angles are equal, each to each, viz. those to which the equal sides are opposite.§**

Let ABC, DEF be two triangles which have the two sides AB,

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sition and in the scholium, will be effected simply by means of the compasses, as was pointed out in the preceding proposition.

* The meaning of the expression *each to each*, or *respectively*, which is used in the same sense, will be known from its application here. Were this expression wanting, the meaning might be merely that the sides AB, AC are together equal to DE, DF; while, when taken separately, they might be either equal or unequal. With the limiting expression, however, the meaning is, that AB is equal to DE, and AC to DF. In such cases, the lines or magnitudes must be taken *in the same order*. Thus, it would be improper in the present case to say, that AB, AC are equal to DF, DE, each to each.

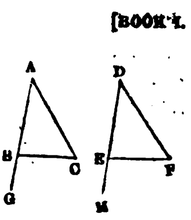
† This expression shows the meaning which Euclid attaches to the *base* of a triangle. It is the *third* side as distinguished from the other two, whether that is the side on which the triangle stands or not.

‡ That is, they have equal areas or surfaces, the *area* of a superficial figure being the space which it contains.

§ This enunciation might be more briefly expressed thus:—

'If two triangles have two sides, and the contained angle of the one respectively equal to two sides and the contained angle of the other; they have likewise their remaining sides equal, and their remaining angles equal, each to each, viz. those which are similarly situated; and their areas are equal.

'In this proposition, the hypothesis is, that there are two triangles which have two sides and the contained angle in one of them, equal respectively to two sides and the contained angle in the other; and it is proved that if this be so, the triangles must be in every respect equal.



AC equal to the two sides, DE, DF, each to each, viz. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF: then (1) the base BC is equal to the base EF; (2) the triangle ABC to the triangle DEF; and (3) the other angles, to which the equal sides are opposite, are equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.

For, if the triangle ABC be applied to DEF, so that the point A may be on D, and the straight line AB upon DE; the point B shall coincide with the point E, because (hyp.) AB is equal to DE; and AB coinciding with DE, AC shall coincide with DF, because (hyp.) the angle BAC is equal to the angle EDF; wherefore also the point C shall coincide with the point F, because (hyp.) AC is equal to DF. But the point B coincides with E; wherefore the base BC shall coincide with the base EF; because, the point B coinciding with E, and C with F, if BC did not coincide with EF, two straight lines would enclose a space, which (I. def. 3, cor.) is impossible. Therefore the base BC shall coincide with the base EF, and (I. ax. 8) be equal to it. Wherefore the whole triangle ABC shall coincide (I. def. 5, cor.) with the whole triangle DEF, and (I. ax. 8) be equal to it; and the other angles of the one shall coincide with the remaining angles of the other, and (I. ax. 8) be equal to them, viz. the angle ABC to DEF, and the angle ACB to DFE.* Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another; their bases are likewise equal, and the triangles are equal, and their other angles to which the equal sides are opposite, are equal, each to each: which was to be demonstrated.

Schol. If the equal sides AB, DE be produced through B and E, the extremities of the bases, the exterior angles CBG, FEH are equal to one another; for the lines which contain them coincide, when the one triangle is applied to the other. In like manner, it would appear that the angles formed by producing AC and DF through C and F are equal.

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* The following observations may assist the beginner in understanding the proof of this important proposition.

When AB is applied to DE, it will coincide with it by the third definition. Also B will coincide with E, because AB is equal to DE. If B fell between D and E, AB would be less than DE; but if B fell on the continuation of DE through E, AB would be greater than DE. Again, the angles A and D are equal; that is, the opening of the lines AB, AC is equal to that of DE and DF. Hence, AB coinciding with DE, AC must coincide with DF. For if AC fell beyond DF, the angle A would be greater than D; but if AC fell between DE and DF, A would be less than D. The point C is then shown to coincide with F, in the same manner in which B was shown to coincide with E. Now, B and C coinciding with E and F, the lines between them must coincide so as to exclude space; for if they did not in every part coincide, a space would be enclosed between them: which (I. def. 3, cor.) is impossible. Hence these lines (I. ax. 8) are equal. The triangles must also coincide entirely: for, since the three sides of the one triangle coincide respectively with those of the other, if the triangles did not coincide internally, a space would be enclosed between two plane surfaces, which (I. def. 5, cor.) is impossible. The triangles are therefore (I. ax. 8) equal. Lastly, since the straight lines BA, BC coincide at the same time with ED, EF, the openings between them must be equal; that is, the angles B and E must be equal: and C and F must be equal for a like reason.

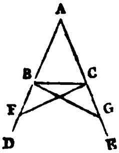
When a proposition is demonstrated by supposing one figure to be applied to another, it is said to be proved by the method of *superposition.*

Prop. V. Theor.—The angles at the base of an isosceles triangle are equal to one another; and, if the equal sides be produced, the angles upon the other side of the base are also equal.*

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the sides AB, AC be produced to D and E; the angle ABC is equal to the angle ACB, and the angle CBD to the angle BCE.

In BD† take any point F, and from AE the greater, cut off (I. 3) AG equal to AF the less, and join (I. post. 1) FC, GB.

Because (const.) AF is equal to AG, and (hyp.) AC to AB, the two sides FA, AC are equal to the two GA, AB, each to each; and they contain the angle A common to the two triangles AFC, AGB: therefore (I. 4, part. 1) the base FC is equal to the base GB; and the remaining angles of the one are equal (I. 4, part. 3) to the remaining angles of the other, each to each, to which the equal sides are opposite; therefore the angle AFC is equal to the angle AGB. And because the whole AF is equal (const.) to the whole AG, of which the parts AB, AC are (hyp.) equal; the remainder BF is equal (I. ax. 3) to the remainder CG; and FC has been proved to be equal to GB: therefore, in the triangles FBC and GCB, the two sides BF, FC are equal to the two CG, GB, each to each: and the angle BFC has been proved to be equal to CGB; wherefore (I. 4, part. 3) their remaining angles are equal, each to each, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and (I. 4, schol.) the angle ABC is equal to the angle ACB, which are the angles at the base of the triangle ABC: and it has been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base. Therefore the angles at the base, &c.; which was to be demonstrated.



Otherwise:—Let the straight line AF‡ divide the angle BAC into two equal parts. Then, in the triangles BAF, CAF, the side BA is equal (hyp.) to CA, and AF is common; therefore the two sides BA, AF, are equal to the two CA, AF, each to each, and the contained angles BAF, CAF are equal: wherefore (I. 4, part. 3) the remaining angles are equal, each to each, to which the equal sides are opposite; therefore, the angle ABF is equal to the angle ACF, and (I. 4, schol.) the exterior angles DBF, ECF, are also equal.

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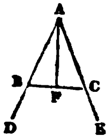
* In this proposition, the hypothesis is, that the triangle under consideration has two of its sides equal; and on this hypothesis it is to be proved that the angles opposite to those sides are equal. The demonstration of this proposition given by Euclid is generally felt by beginners to be one of the most difficult in the Elements. The one here given proceeds on the same general principle, but is shorter and easier.

† The proof would be equally easy were F and G taken in AB and AC.

‡ Through the point A an infinite number of straight lines may be drawn, and it is plain that there is one of these which divides the angle BAC into two equal parts. In assuming the truth of this, however, we virtually employ an additional

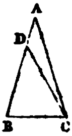


Cor. Hence every equilateral triangle is also equiangular. This is shown by taking, first, one side as base, and then another.

PROP. VI. THEOR.*—If two angles of a triangle be equal to one another, the sides which subtend, *or are opposite to,* those angles, are also equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the side AB is also equal to the side AC.

For, if AB be not equal to AC,† one of them is greater than the other: let AB be the greater, and from it cut off (I. 3) BD equal to AC, the less, and join DC. Then, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two sides DB, BC are equal to the two AC, CB, each to each; and (hyp.) the angle DBC is equal to the angle ACB; therefore (I. 4, part 2) the triangle DBC is equal to the triangle ACB, the less to the greater; which is absurd. Therefore AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles, &c.



Cor. Hence every equiangular triangle is also equilateral.

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axiom, which ought not to be done without a good reason. In other respects this proof is perfectly satisfactory; and on account of its ease and simplicity, it may perhaps be preferred by those who are reading the Elements for the first time.

This proposition may also be proved by conceiving the triangle BAC to be inverted, and to be applied to the space which it before occupied, so that the point A may retain its position, while AC will fall on AB; then will the point C fall on B, AB on AC, and B on C; and the angles ACB, ECB will coincide with ABC, DBC respectively, and (I. ax. 8) be equal to them.

It may be remarked that Euclid's proof, when stripped of artifice, and referred to the principle of superposition, on which it, and every demonstration which has the fourth proposition for its basis, ultimately depends, is reduced to an inversion and mutual application such as that indicated above. This will appear evident from proving the equality of the triangles ABG, ACF by superposition.

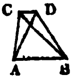
* This proposition is the converse of the fifth; that is, in the language of logic, the subject of this is the predicate of the fifth, and the predicate of this the subject of the fifth. Thus, it is proved in the fifth, that if two sides of a triangle be equal, the angles opposite to them are also equal; and in the sixth, that if two angles of a triangle be equal, the sides opposite to them are equal.

† This demonstration is indirect. To prove AB equal to AC, we suppose them, if possible, to be unequal. Then one of them, suppose AB, being considered the greater, we cut off a part DB, terminated at one of the equal angles, which we assume as equal, if possible, to AC; and joining CD, we prove, by means of the fourth proposition, that if the supposition were true, the triangle DBC would be

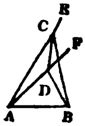
Prop. VII. Theor.*—Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, let there be two triangles ABC, ABD, upon the same base AB, and upon the same side of it, which have their sides CA, DA, terminated in the extremity A of the base equal to one another, and likewise their sides CB, DB, that are terminated in B.

Join CD; then, in the case in which the vertex of each of the triangles is without the other triangle, because AC is equal to AD, the angle ACD is equal (I. 5) to the angle ADC. But the angle ACD is greater (I. ax. 9) than BCD; therefore the angle ADC is greater also than BCD; much more then is the angle BDC greater than BCD. Again, because CB is equal to DB, the angle BDC is equal (I. 5) to BCD; but it has been demonstrated to be greater than it; therefore the supposition, that AC is equal to AD, and likewise BC to BD, is false.



But if one of the vertices, as D, be within the other triangle ACB; produce AC, AD to E, F; therefore, because AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other side of the base CD are equal (I. 5) to one another; but the angle ECD is greater (I. ax. 9) than BCD; wherefore the angle FDC is likewise greater than BCD: much more then is BDC greater than BCD. Again, because CB is equal to DB, the angle BDC is equal (I. 5) to BCD; but BDC has been proved to be greater than the same BCD; and therefore AC cannot be equal to AD, and at the same time BC to BD. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration. Therefore on the same base, and on the same side of it, &c.



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equal in magnitude to ACB: but this being contrary to the 9th axiom, we conclude that AB and AC cannot be unequal, since the supposition that they are so, leads to an absurd conclusion. The student may exercise himself usefully in proving this proposition by producing CA through the vertex, and by supposing the whole line thus produced to be equal, if possible, to AB: and he may in like manner have two variations of the proof by supposing AC greater than AB.

It would perhaps remove, in some degree, the difficulty which beginners often feel with respect to negative demonstrations, were they expressed hypothetically. Thus, in the present instance, we might proceed briefly in the following manner:—For, if AB were not equal to AC, one of them, as AB, would be the greater. Then some part of AB, as BD, would be equal to AC; and, CD being joined, there would be two triangles DBC, ACB, in which there would be two sides and the contained angle of the one respectively equal to two sides and the contained angle of the other, and therefore (I. 4) the triangle DBC would be equal to ACB, which (I. ax. 9) is impossible. Therefore AB is not unequal to AC, that is, it is equal to it.

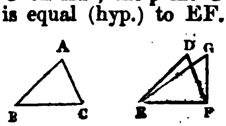
It is often stated that this proposition affords the first instance in which the indirect method of proof is employed. This, however, is incorrect, as it is on this principle that the bases and the triangles themselves are proved to coincide in the fourth proposition, by showing that their failing to coincide would be contrary to the corollaries to the 3d and 5th definitions.

* This proposition is merely a lemma to the 8th, and may be omitted, if the second method of demonstrating that proposition be adopted.

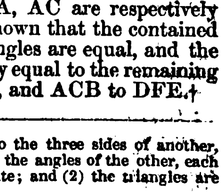
Prop. VIII. Theor.—If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal: (1.) the angle which is contained by the two sides of the one is equal to the angle contained by the two sides, equal to them, of the other; and (2.) the two triangles are equal to one another.*

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and also the base BC equal to the base EF; the angle BAC is equal to the angle EDF, and the two triangles are equal in area.

For, if the triangle ABC be applied to DEF, so that the point B may be on E, and the straight line BC on EF; the point C shall also coincide with F, because BC is equal (hyp.) to EF. Therefore, BC coinciding with EF, BA and AC shall coincide with ED and DF; for, if the base BC coincide with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation, as EG, FG; then, upon the same base EF, and upon the same side of it, there would be two triangles having their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity; but (I. 7) this is impossible: therefore, if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF; wherefore also the angle BAC coincides with the angle EDF, and (I. ax. 8) is equal to it. Also (I. def. 5, cor.) the triangle ABC coincides with the triangle DEF, and (I. ax. 8) is equal to it. Therefore, if two triangles, &c.



Otherwise:—Let the triangle ABC be inverted with respect to the base BC, and let BC be applied to EF, so that B may fall on E; then will C coincide with F, because BC is equal to EF; and the triangle ABC will take the position GEF, EG being the same as AB, and FG the same as CA. Join DG; and, because (hyp.) DE and EG are equal, the angles EDG, EGD are (I. 5) equal. It would be shown in a similar manner, that the angles FDG, FGD are equal; and, therefore, (I. ax. 2) the angle EGF, that is, BAC, is equal to EDF. But (hyp.) the sides BA, AC are respectively equal to ED, DF, and it has now been shown that the contained angles are equal; therefore (I. 4) the triangles are equal, and the remaining angles in the one are respectively equal to the remaining angles in the other, that is, ABC to DEF, and ACB to DFE.†



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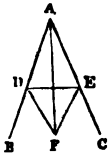
* Or, if the three sides of one triangle be equal to the three sides of another, each to each; (1) the angles of the one are equal to the angles of the other, each to each, viz. those to which the equal sides are opposite; and (2) the triangles are equal to one another.

† Should DG fall on DF and FG, or on DE and EG, the proof would be had by

PROP. IX. PROB.—To bisect a given rectilineal angle, that is, *to divide it into two equal angles.*

Let BAC be the given angle; it is required to bisect it.

Take any point D in AB, and from AC cut off (I. 3) AE equal to AD; join DE, and upon it describe (I. 1) an equilateral triangle DEF, on the side remote from A;* then join AF: AF bisects the angle BAC.

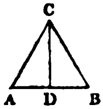


Because AD is equal (const.) to AE, and AF is common to the two triangles DAF, EAF; the two sides DA, AF are equal to the two EA, AF, each to each; and the base DF is equal (const. and I. def. 17) to the base EF: therefore, the angle DAF is equal (I. 8) to EAF; wherefore, the given angle BAC is bisected by the line AF: which was to be done.†

PROP. X. PROB.‡—To bisect a given finite straight line.

Let AB be the given line: it is required to bisect it.

Describe (I. 1) upon it an equilateral triangle ABC, and bisect (I. 9) the angle ACB by the straight line CD: AB is bisected in the point D.§



Because AC is equal to CB, and CD common to the two triangles ACD, BCD; the two sides AC, CD, are equal to BC, CD, each to each; and the angle ACD is equal (const.) to the angle BCD; therefore the base AD is equal (I. 4) to the base DB, and the line AB is bisected in the point D: which was to be done.

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means of one isosceles triangle. But should the point F fall within the isosceles triangle DEG, or E within DFG, two isosceles triangles would still be requisite, and the equality of the vertical angles would be proved by means of the third axiom.

[*] The expression, "on the side remote from A," is added in this edition, because, if the equilateral triangle were described on the other side, its vertex might fall on the vertex of the angle to be bisected (as it would do in the next proposition), in which case the solution would fail, as no second point in the bisecting line would then be determined.

In the practical construction of this problem, arcs may be described from D and E as centres with any radius greater than the half of DE, and the line joining either of their points of intersection with A will bisect the given angle. It is plain, that if the radius be taken equal to AD or AE, A must be joined with the remote point of intersection, as A and the other would coincide.

If the angles BAF, CAF were bisected, the given angle would be divided into four equal parts; the bisection of which would divide it into eight equal parts; and thus the division, by successive bisections, might be continued without limit.

[†] Otherwise.—Let DAE (1st fig. to prop. 5) be the given angle; in AD assume any points, B and F, and cut off AC and AG, respectively equal to AB and AF: join BG and CF, and the straight line joining their point of intersection with A bisects the angle. The proof, which is easy, is left to exercise the learner.

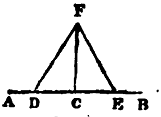
[‡] This problem is a particular case of the ninth proposition of the sixth book.

[§] In practice, the construction is effected more easily by describing arcs on both sides of AB, from A as centre, and with any radius greater than the half of AB; and then, by describing arcs intersecting them, with an equal radius, from B as centre: the line joining the two points of intersection will bisect AB. The proof is easy.

Prop. XI. Prob.*—To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a point given in it; it is required to draw a straight line from C at right angles to AB.

Take any point D in AC, and make (I. 3) CE equal to CD; upon DE describe (I. 1) the equilateral triangle DFE, and (I. post. 1) join FC; FC is the required line.†

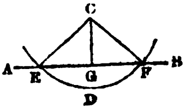


Because DC is equal to CE, and FC common to the two triangles DCF, ECF; the two sides DC, CF are equal to the two EC, CF, each to each; and the base DF is equal (const. and I. def. 17) to the base EF; therefore, the angle DCF is equal (I. 8) to the angle ECF; and they are adjacent angles. But, when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called (I. def. 8) a right angle: therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given straight line AB, FC has been drawn at right angles to AB: which was to be done.

Prop. XII. Prob.—To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced any length both ways, and let C be a point without it. It is required to draw a straight line from C perpendicular to AB.

Take any point D upon the other side of AB, and from the centre C, at the distance CD, describe (I. post. 3) the circle EDF meeting AB in E and F: bisect (I. 10) EF in G, and join CG; the straight line CG is the perpendicular required.‡



Join CE, CF. Then, because EG is equal to GF, and CG common to the triangles EGC, FGC, the two sides, EG, GC are equal to the two, FG, GC, each to each; and the base CE is equal (I. def. 30) to the base CF; therefore the angle CGE is equal (I. 8) to the angle CGF; and they are adjacent angles: therefore CG is perpendicular (I. def. 8) to AB. Hence, from the given point C a perpendicular CG has been drawn to the given line AB: which was to be done.

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* This proposition and the following contain the only two distinct cases of drawing a perpendicular to a given straight line through a given point; the first, when the point is *in* the line, the second, when it is *without* it.

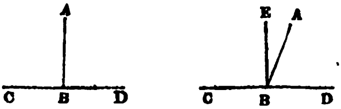
† In the practical construction of this problem, it is sufficient to describe arcs from D and E as centres, with any radius greater than DC, and to join their point of intersection with C. The proof follows from the construction and the eighth proposition. It will be a check on the manual operation, if intersections be made on both sides of AB, as these intersections and C should all be in the same straight line, as will appear from the corollary to the 14th proposition of this book.

‡ In practice the construction will be made rather more simple, by describing from F and E, when found, arcs on the remote side of AB from C, with any radius greater than the half of FE, and joining their point of intersection with C.

The 12th proposition would be better placed after the 16th; as we cannot prove, without the assistance of the 16th, that there can be but one perpendicular drawn to a straight line from a point without it. We might then infer also, that a circle can cut a straight line in only two points, since if it could cut it in more than two, there might be more than one perpendicular.

Prop. XIII. Theor.—The angles which one straight line makes with another upon the one side of it are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD; these are together equal to two right angles.



For, if the angle CBA be equal to ABD, each of them (I. def. 8) is a right angle. But if not, from the point B draw (I. 11) BE at right angles to CD;[*] therefore the angles CBE, EBD, are two right angles. Then (I. ax. 10) DBE is equal to the two angles DBA, ABE together; add EBC to each of these equals: therefore the angles CBE, EBD, are equal (I. ax. 2) to the three angles DBA, ABE, EBC. Again (I. ax. 10), the angle CBA is equal to the two angles CBE, EBA; add ABD to these equals: therefore the angles CBA, ABD are equal to the three angles CBE, EBA, ABD. But CBE, EBD have been demonstrated to be equal to the same three angles: therefore (I. ax. 1) the angles CBE, EBD are equal to the angles CBA, ABD; but CBE, EBD are two right angles: therefore CBA, ABD are together equal to two right angles. Wherefore the angles which one straight line, &c.

Cor. 1. From this it is manifest, that if two straight lines cut one another, the four angles which they make at their point of intersection, are together equal to four right angles.

Cor. 2. And consequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

Prop. XIV. Theor.—If, at a point in a straight line, two other straight lines on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles

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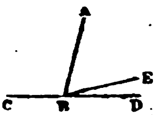
[*] The substance of the remainder of this demonstration is, that the angles ABC, ABD are together equal to EBC, EBD, each pair being equal to the three angles CBE, EBA, ABD.

ABC, ABD equal together to two right angles: BD is in the same straight line with CB.

For if BD be not in the same straight line with CB, let BE be in the same straight line with it. Therefore, because AB makes angles with the straight line CBE, upon one side of it, those angles ABC, ABE, are together equal (I. 13) to two right angles; but the angles ABC, ABD are likewise together equal (hyp.) to two right angles: therefore (I. ax. 11 and 1) the angles CBA, ABE are equal to the angles CBA, ABD: take away the common angle ABC, and the remaining angle ABE is equal (I. ax. 3) to the remaining angle ABD, a part to the whole, which is absurd. Therefore BE is not in the same straight line with BC: and in like manner it may be demonstrated, that no other line can be in the same straight line with it but BD, which therefore is in the same straight line with CB. Wherefore, if at a point, &c.



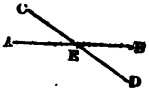
Cor. If, at a point in a straight line, two other straight lines meet on the opposite sides of it, and make equal angles with the parts of it on opposite sides of the point, the two straight lines are in one and the same straight line.

Let AEB (fig. to the next proposition) be a straight line, and let the angles AEC, BED be equal; CE, ED are in the same straight line. For, by adding the angle CEB, to the equal angles AEC, BED, we have BED, BEC together equal to AEC, CEB, that is (I. 13) to two right angles; and therefore, by this proposition, CE, ED are in the same straight line.

PROP. XV. THEOR.—If two straight lines cut one another, the vertical, *or opposite* angles are equal.

Let the two straight lines AB, CD cut one another in E; the angle AEC is equal to the angle DEB, and CEB to AED.

Because the straight line AE, makes with CD the angles CEA, AED, these angles are together equal (I. 13) to two right angles. Again, because DE makes with AB the angles AED, DEB, these also are together equal to two right angles; and CEA, AED have been demonstrated to be equal to two right angles; wherefore (I. ax. 11 and 1) the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle AED, and (I. ax. 3) the remaining angles CEA, DEB are equal.* In the same manner it can be demonstrated that the angles CEB, AED are equal. Therefore, if two straight lines, &c.



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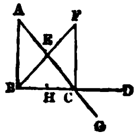
* In the proof here given, the common angle is AED; and CEB might with equal propriety be made the common angle. In like manner, in proving the equality of CEB and AED, either AEC or BED may be made the common angle. It is also evident, that when AEC and BED have been proved to be equal, the equality of AED and BEC might be inferred from the 13th proposition, and the third axiom.

PROP. XVI. THEOR.—If one side of a triangle be produced, the exterior angle is greater than either of the interior remote angles.

Let ABC be a triangle, and let its side BC be produced to D; the exterior angle ACD is greater than either of the interior remote angles CBA, BAC.

Bisect (I. 10) AC in E, join (I. post. 1) BE, and produce it (I. post. 2) to F; make (I. 3) EF equal to BE; and join FC.



Because (const.) AE is equal to EC, and BE to EF; the two sides, AE, EB are equal to the two, CE, EF, each to each; and the angles AEB, CEF are equal (I. 15), because they are opposite, vertical angles: therefore the angle BAE is equal (I. 4, part 3) to ECF; but the angle ECD is greater (I. ax. 9) than ECF: therefore the angle ACD is greater than BAE.

In the same manner, if the side AC be produced (I. post. 2) to G, and BC be bisected in H, and if a straight line be drawn from A to H, and continued till the produced part be equal to AH, and if the extremity of that part be joined with C, it may be demonstrated that the angle BCG, that is (I. 15) ACD is greater than ABC. Therefore, if one side, &c.

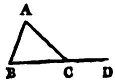
Cor. 1. Hence from a point without a straight line, only one perpendicular can be drawn to the line. For, if there could be two, a triangle would be formed which would have an exterior angle equal to an interior and remote one, each being a right angle; which, by the proposition, is impossible.

Cor. 2. If from a point without a line, two straight lines be drawn cutting it, one of them at right angles to it, and the other not, the perpendicular one will fall on that side of the other on which the acute angle is: for if it fell on the other, the exterior acute angle would be greater than the interior and remote right angle, which (I. def. 10) is impossible.

PROP. XVII. THEOR.*—Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle: any two of its angles are together less than two right angles.

Produce BC to D. Then, because ACD is the exterior angle of the triangle ABC, it is greater (I. 16) than the interior and remote angle ABC. To each of these add ACB: therefore the angles ACD, ACB are greater (I. ax. 4) than ABC, ACB: but ACD, ACB are together equal (I. 13) to two right angles: therefore the angles ABC, BCA are less than two right



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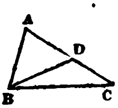
* This proposition might be omitted, as it is included in the 32d of this book, and is not referred to in any thing preceding that proposition.

angles. In like manner it may be demonstrated, that BAC, ACB are less than two right angles; and if CB be produced through B, the same may be demonstrated respecting the angles A and B. Therefore any two angles, &c.

Cor. Hence every triangle must have at least two acute angles.

PROP. XVIII. THEOR.*—If two sides of a triangle be unequal, the greater side has the greater angle opposite to it.

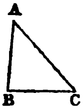
Let ABC be a triangle, of which the side AC is greater than the side AB; the angle ABC is also greater than the angle BCA.



Because AC is greater than AB, make (I. 3) AD equal to AB, and join BD; and because ADB is the exterior angle of the triangle BDC, it is greater (I. 16) than the interior and remote angle C; but (I. 5) ADB is equal to ABD, because the sides AB, AD are equal: therefore the angle ABD is likewise greater than C; wherefore much more is the angle ABC greater than C. Therefore, if two sides, &c.

PROP. XIX. THEOR.—If two angles of a triangle be unequal, the greater angle has the greater side opposite to it.

Let ABC be a triangle, of which the angle B is greater than C; the side AC is likewise greater than AB.



For if it be not greater, AC must either be equal to AB, or less than it. It is not equal, because then (I. 5) the angle B would be equal to the angle C; but it is not. Neither is it less; because then (I. 18) the angle B would be less than the angle C, but it is not. AC, therefore, is not less than AB; and it has been shown that it is not equal to AB: therefore AC is greater than AB. Wherefore, if two angles, &c.

PROP. XX. THEOR.—Any two sides of a triangle are together greater than the third side.†

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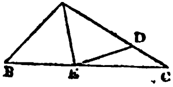
* Let the learner compare this proposition and the following with the 5th and 6th of this book.

This proposition might be proved with equal facility by producing AB through B, till the whole line thus produced is equal to AC, and joining C with the extremity of the produced part. The following proof is very neat and easy.



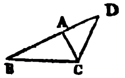
From AC cut off AD equal to AB; bisect (I. 9) the angle BAC by the straight line AE, and join ED. Then, in the triangles BAE, DAE, BA is equal to DA, AE common, and the angle BAE equal to DAE: therefore (I. 4, part 3) the angle ADE is equal to B. But (I. 16) ADE is greater than C; therefore also B is greater than C.

† The truth of this proposition is so manifest, that a proof of it is given merely to avoid increasing the number of axioms. If, as is done by Archimedes, a straight line were defined to be the shortest distance between two points, this proposition would follow immediately as a corollary.

Let ABC be a triangle; any two sides of it are together greater than the third side.

Produce (I. post. 2) BA to the point D, making (I. 3) AD equal to AC; and join DC.

Because DA is equal to AC, the angle ADC is likewise equal (I. 5) to ACD; but the angle BCD is greater (I. ax. 9) than ACD: therefore the angle BCD is greater than ADC: and because in the triangle DBC, the angle BCD is greater than BDC, and since (I. 19) the greater side is opposite to the greater angle: therefore the side DB is greater than the side BC; but DB is equal to BA and AC; therefore BA, AC are greater than BC. In the same manner, it might be demonstrated, that the sides AB, BC are greater than CA, and BC, CA greater than AB. Therefore, any two sides, &c.

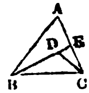


Cor. Hence the difference of any two sides of a triangle, is less than the third side. For, since BA and AC are together greater than BC, from each of these take AC; then (I. ax. 5) BA is greater than the excess of BC above AC.

PROP. XXI. THEOR.*—If from a point within a triangle, two straight lines be drawn to the extremities of one of the sides, these lines are together less than the other sides, but contain a greater angle.

From the point D, within the triangle BAC, let the straight lines DB, DC be drawn to the extremities of BC; DB, DC are less than AB, AC, but the angle BDC is greater than BAC.

Produce (I. post. 2) BD to E. Then, because two sides of a triangle are greater (I. 20) than the third, the two sides BA, AE are greater than BE. To each of these add EC: therefore (I. ax. 4) BA, AC are greater than BE, EC. Again, because (I. 20) the two sides CE, ED are greater than CD, add DB to each: therefore CE, EB are greater than CD, DB. But it has been shown that BA, AC are greater than BE, EC: much more then are BA, AC greater than BD, DC.



Again, the exterior angle BDC of the triangle CDE is greater (I. 16) than CED; for the same reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle BDC is greater than CEB: much more then is the angle BDC greater than BAC. Therefore, if from a point, &c.

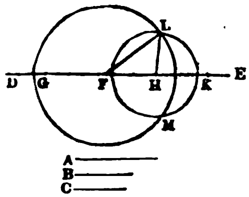
A proof as easy as the one given above would be obtained by bisecting the angle BAC; as it follows from the 16th proposition, that each of the angles made by the bisecting line with BC is greater than the half of BAC; and therefore (I. 19) each segment of BC is less than the adjacent side.

* This proposition is never referred to by Euclid, except in the 8th proposition of the third book; and that proposition may be proved without it.

By means of the 32d of this book, it may be proved that the angle BDC exceeds BAC by the sum of the angles ABD, ACD.

PROP. XXII. PROB.—To describe a triangle of which the sides shall be equal to three given straight lines; but any two of these must be greater (I. 20) than the third.

Let A, B, C be three given straight lines, of which any two are greater than the third: it is required to make a triangle of which the sides shall be equal to A, B, C, each to each.



Take an unlimited straight line DE, and let F be a point in it, and make (I. 3) FG equal to A, FH to B, and HK to C. From the centre F, at the distance FG, describe (I. post. 3) the circle GLM, and from the centre H, at the distance HK, describe the circle KLM. Now, because (hyp.) FK is greater than FG, the circumference of the circle GLM, will cut FE between F and K, and therefore the circle KLM cannot lie wholly within the circle GLM. In like manner, because (hyp.) GH is greater than HK, the circle GLM cannot lie wholly within the circle KLM. Neither can the circles be wholly without each other, since (hyp.) GF and HK are together greater than FH. The circles must therefore intersect each other: let them intersect in the point L, and join LF, LH; the triangle LFH has its sides equal respectively to the three lines A, B, C.

Because F is the centre of the circle GLM, FL is equal (I. def. 30) to FG; but (const.) FG is equal to A: therefore (I. ax. 1) FL is equal to A. In like manner it may be shown that HL is equal to C; and (const.) FH is equal to B: therefore the three straight lines LF, FH, HL are respectively equal to the three A, B, C: and therefore the triangle LFH has been constructed, having its three sides equal to the three given lines, A, B, C: which was to be done.*

PROP. XXIII. PROB.—At a given point in a given straight line, to make a rectilineal angle equal to a given one.

Let AB be the given straight line, A the given point in it, and C the given angle: it is required to make an angle at A, in the straight line AB, that shall be equal to C.

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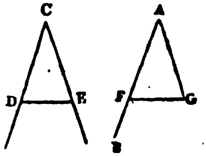
* It it evident that if MF, MH were joined, another triangle would be formed, having its sides equal to A, B, C. It is also obvious that in the practical construction of this problem, it is only necessary to take with the compasses FH equal to B, and then, the compasses being opened successively to the lengths of A and C, to describe circles or arcs from F and H as centres, intersecting in L; and lastly to join LF, LH.

The construction in the text is made somewhat different from that given in Simson's Euclid, with a view to obviate objections arising from the application of this proposition in the one that follows it.

In the lines containing the angle C, take any points D, E, and join them, and make (I. 22) the triangle AFG, the sides of which AF, AG, FG, shall be equal to the three straight lines CD, CE, DE, each to each. Then, because FA, AG are equal to DC, CE, each to each, and the base FG to the base DE, the angle A is equal (I. 8) to the angle C. Therefore, at the given point A in the given straight line AB, the angle A is made, equal to the given angle C: which was to be done.[*]

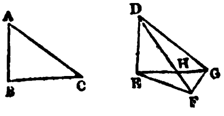


Prop. XXIV. Theor.—If two triangles have two sides of the one equal to two sides of the other, each to each, but the angles contained by those sides unequal: the base of that which has the greater angle is greater than the base of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two DE, DF, each to each; but the angle BAC greater than EDF: the base BC is also greater than the base EF.

Of the two sides DE, DF, let DE be that which is not greater than the other, and at the point D, in the straight line DE, make (I. 23) on the same side with EDF the angle EDG equal to BAC; make also DG equal (I. 3) to AC or DF, and join EG, GF.[†]

Because AB is equal to DE, and AC to DG, the two sides BA, AC are equal to the two ED, DG, each to each; and the angle BAC is equal to the angle EDG: therefore the base BC is equal (I. 4) to the base EG. Again, because DG is equal to DF, the angles (I. 5) DFG, DGF are equal; but the angle DGF is greater



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[*] The construction of this proposition is most easily effected in practice, by making the triangles isosceles. In doing this, circular arcs are described with equal radii from C and A as centres, and their chords are made equal.

It is evident that another angle might be made at A, on the other side of AB, equal to C.

[†] Hence the angle DGE is not greater (I. 5, and 18) than DEG; but DHG is greater (I. 16) than DEG: therefore DHG is greater than DGH, and (I. 19) DG or DF is greater than DH; and consequently the point H lies between D and F, and the line EG above EF. If, on the contrary, a line equal to DE the less side, were drawn through D, making with DF, on the same side of it with DE, an angle equal to A, the extremity of that line might have fallen on FE produced, or above it or below it; and thus the proof would require three cases, while the method here given, which is that of Simson, is universally applicable.

We should also have an easy proof by bisecting the angle FDG by a straight line cutting EG in a point, which call K, and joining DK and KF. For KG would be equal (I. 4) to KF; and by adding EK, we should have EG equal to EK, KF, and, therefore (I. 20) EG greater than EF.

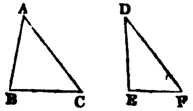
Let the student compare this proposition and the following with the fourth and eighth of this book.

(I. ax. 9) than EGF: therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF. Then, because the angle EFG of the triangle EFG is greater than its angle EGF; the side EG is greater (I. 19) than the side EF: but EG is equal to BC; and therefore also BC is greater than EF. Therefore, if two triangles, &c.

PROP. XXV. THEOR.—If two triangles have two sides of the one equal to two sides of the other, each to each, but their bases unequal; the angle contained by the sides of that which has the greater base, is greater than the angle contained by the sides equal to them, of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two DE, DF, each to each; but the base BC greater than the base EF: the angle A is likewise greater than the angle D.

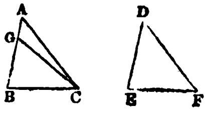
For, if it be not greater, it must either be equal to it or less; but the angle A is not equal to D, because then (I. 4) the base BC would be equal to EF; but it is not. Neither is it less; because then (I. 24) the base BC would be less than EF; but it is not: therefore the angle A is not less than the angle D; and it has been shown, that it is not equal to it: therefore, the angle A is greater than D. Wherefore, if two triangles, &c.



PROP. XXVI. THEOR.—If two angles of one triangle be equal to two angles of another, each to each, and if a side of the one be equal to a side of the other similarly situated with respect to those angles; (1) the remaining sides are equal, each to each; (2) the remaining angles are equal; and (3) the triangles are equal to one another.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, viz. ABC to DEF, and BCA to EFD; also one side equal to one side: and,

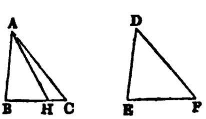
First, let those sides be equal to which the angles are adjacent, that are equal in the two triangles; viz. BC to EF: the other sides are equal, each to each, viz. AB to DE, and AC to DF; and the remaining angles BAC, EDF are equal; also the areas of the triangles are equal.



For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater, and make (I. 3) BG equal to DE, and join GC; then because GB is equal to DE, and (hyp.) BC to EF, the two sides GB, BC are equal to the two DE, EF, each to each; and (hyp.) the angle GBC is equal to the angle DEF : therefore the angle GCB is equal (I. 4, part. 3) to DFE. But DFE is (hyp.) equal to ACB: wherefore, also the angle GCB is equal to ACB, a part to the whole, which (I. ax. 9) is impossible: therefore AB is not unequal to DE, that it, it is equal to it. Then, since BC is equal to EF, the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC to DEF: therefore (I. 4) the base AC is equal to the base DF, the triangle ABC to the triangle DEF, and the third angle BAC to the third angle EDF.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, viz. AB to DE; likewise in this case the other sides are equal, viz. BC to EF, and AC to DF; also the triangle ABC to DEF, and the third angle BAC to the third angle EDF.

For, if BC be not equal to EF, let BC be the greater, and make (I. 3) BH equal to EF, and join AH. Then, because BH is equal to EF, and (hyp.) AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles: therefore (I. 4, part 3) the angle BHA is equal to EFD; but EFD is equal (hyp.) to BCA: therefore also the angle BHA is equal to BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and remote angle BCA, which (I. 16) is impossible: wherefore BC is not unequal to EF, that is, it is equal to it. Then, since AB is equal (hyp.) to DE, and the angle B to the angle E: therefore (I. 4) the base AC is equal to the base DF, the third angle BAC to the third angle EDF, and the triangle ABC to the triangle DEF.* Therefore, if two triangles, &c.



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* By the fifth corollary to the 32d proposition of this book, if two angles of one triangle be equal to two angles of another, their remaining angles are also equal. This proposition, therefore, if it be postponed till the student has learned the 32d, may be demonstrated in a single case, the same as either the first or second in Euclid's method, given above; and it may be so postponed without impropriety, as the 32d does not depend on it, either directly or indirectly. If the proposition be taken thus, the enunciation may be as follows :—

If two triangles be equiangular to one another, and if a side of the one and a side of the other, which are opposite to equal angles, be equal; then (1) *the remaining sides are equal, each to each, viz. those which are opposite to equal angles; and* (2) *the triangles are equal.*

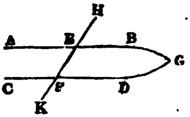
This is the third of the propositions in which triangles are demonstrated to be in every respect equal. They are proved to be such in the fourth proposition, when two sides and the contained angle of the one are respectively equal to two sides and the contained angle of the other: in the eighth, when the three sides of the one are equal to the three sides of the other, each to each: and in the twenty-sixth, when two angles and a side of the one are respectively equal to two angles and a side, similarly situated, of the other. There is only one other case in which two triangles are in all respects equal; that is, when a side and the opposite angle of one of them are respectively equal to a side and the opposite angle of the other, and a second side in the one equal to a second side in the other, the angles opposite to these latter sides being of the same kind; that is, both right angles, both acute, or both obtuse. This proposition, which is a case of the 7th of the sixth book, is of little use, except that it completes the theory; but it may be a proper exercise for the student to prove it.

It will be seen from these remarks, that two triangles are in every respect equal, when of the sides and angles any three, except the three angles in one of them, are

Prop. XXVII. Theor.—If a straight line falling upon two other straight lines, in the same plane, make the alternate angles equal to one another, these two straight lines are parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, in the same plane, make the alternate angles, AEF, EFD, equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD, being produced, will meet either towards B, D, or towards A, C: let them be produced and meet towards B, D, in the point G. Therefore GEF is a triangle, and its exterior angle AEF is greater (I. 16) than the interior and remote angle EFG; but (hyp.) it is also equal to it, which is impossible: therefore, AB and CD, being produced, do not meet towards B, D. In like manner, it may be demonstrated, that they do not meet towards A, C. But those straight lines which are in the same plane, and which meet neither way, though produced ever so far, are parallel (I. def. 11) to one another: AB therefore is parallel to CD.* Wherefore, if a straight line, &c.



Prop. XXVIII. Theor.—Two straight lines are parallel to one another, (1) if a straight line falling upon them make the exterior angle equal to the interior and remote, upon the same side of that line; and (2) if it make the interior angles upon the same side together equal to two right angles.

Let the straight line EF, which falls upon the two straight lines, AB, CD, make the exterior angle EGB equal to the interior and remote angle GHD upon the same side of EF, or make the interior angles on the same side, BGH, GHD, together equal to two right angles; AB is in each case parallel to CD.

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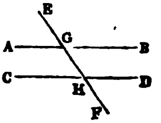
respectively equal to the corresponding sides or angles, in the other, with only the one limitation in the proposition just enunciated, that in it the angles opposite to two equal sides must be of the same kind: and from this it appears, that of the sides and angles of a triangle, three must be given to determine the triangle, and these three cannot be the three angles. Were only the three angles given, the sides, as will appear from the 32d proposition of this book, might be of any magnitudes whatever.

The student may exercise himself in proving this proposition by superposition; and also by producing BA and BC, instead of cutting off parts of them.

* In this proof ABG and CDG must be regarded as straight lines. It is not necessary, indeed, to make the actual construction here given; for the student will see that AB and CD cannot meet on either side, since, if they did, a triangle would thus be formed, and the exterior angle would be (I. 16) greater than the interior and remote angle, which is contrary to the hypothesis. The *alternate angles* as they are understood by Euclid, or the *interior alternate angles*, as they might be called with perhaps more propriety, are the interior remote angles on opposite sides of the line which falls on the other two. AEH and KFD, and also BEH and CFK, may be called *exterior alternate angles*; and it is easy to prove that if these be equal, the lines are parallel.

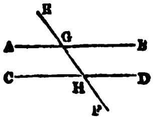
Because the angle EGB is equal (hyp.) to GHD, and EGB (I. 15) to AGH, the angle AGH is equal (I. ax. 1) to GHD; and they are alternate angles: therefore AB is parallel (I. 27) to CD.



Again, because the angles BGH, GHD are equal (hyp.) to two right angles; and that AGH, BGH are also equal (I. 13) to two right angles; the angles AGH, BGH are equal (I. ax. 1 and 11) to BGH, GHD; take away the common angle BGH; therefore the remaining angles AGH, GHD are equal; and they are alternate angles; therefore AB is parallel (I. 27) to CD.* Wherefore two straight lines, &c.

PROP. XXIX. THEOR.—If a straight line fall upon two parallel straight lines, (1) it makes the alternate angles equal to one another; (2) the exterior angle equal to the interior and remote upon the same side; and (3) the two interior angles upon the same side together equal to two right angles.†

Let the straight lines AB, CD be parallel, and let EF fall upon them; then (1) the alternate angles AGH, GHD are equal to one another; (2) the exterior angle EGB is equal to the interior and remote, upon the same side, GHD; and (3) the two interior angles BGH, GHD upon the same side are together equal to two right angles.



For, if AGH be not equal to GHD, one of them must be greater than the other; let AGH be the greater, and to each of them add the angle BGH: therefore the angles AGH, BGH are greater (I. ax. 4) than the angles BGH, GHD; but the angles AGH, BGH are equal (I. 13) to two right angles: wherefore the angles BGH, GHD are less than two right angles. But (I. ax. 12) those straight lines, which with another straight line falling upon them, make the interior angles on the same side less than two right angles, meet together, if continually produced: therefore the straight lines AB, CD, if produced far enough, will meet: but (I. def. 11) they never meet, since (hyp.) they are parallel: therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it. But the angle AGH is equal (I. 15) to the angle EGB: therefore, likewise EGB is equal (I. ax. 1) to GHD. Add to each of these the angle BGH: therefore the angles EGB, BGH are equal (I. ax. 2) to the angles BGH, GHD; but EGB, BGH are equal (I. 13) to two right angles: therefore also BGH, GHD are equal to two right angles. Wherefore, if a straight line, &c.

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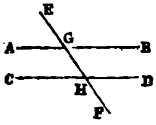
* It is easy to prove, that if the *exterior* angles on the same side of the line which falls on the other two, be together equal to two right angles, the straight lines are parallel.

This proposition will readily lead us to admit the truth of what is called the 12th axiom. It is here proved, that if the two angles BGH, GHD be together equal to two right angles, AB and CD are parallel, and consequently do not meet. Now, if through G a straight line be drawn in the angle BGH, and consequently making with GH an angle less than BGH, so that the two angles which GH makes with it and with HD are together less than two right angles, we will readily admit, that if that line and HD be continually produced, they will at length meet: which is the 12th axiom. The truth of this axiom, therefore, will be admitted by any one who has read the Elements thus far, but not, as is the case, with regard to the other axioms, by a person who is unacquainted with the principles of geometry.

† The first part of this proposition is the converse of the 27th, and the second and third parts are the converses of the first and second parts of the 28th.



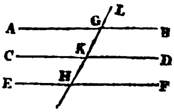
Cor. 1. Hence, if there be two parallels, a straight line which intersects one of them, will intersect the other, if continually produced. Thus, if any straight line be drawn through G, intersecting AB, the angles which it and CD make on one side with a straight line drawn from G to any point H in CD, will be together less than two right angles, and therefore (I. ax. 12) the lines will meet.

Cor. 2. Hence, also, two straight lines which intersect each other are not both parallel to the same straight line.

PROP. XXX. THEOR.—Two straight lines which are not in the same straight line, and which are parallel to a third straight line, are parallel to one another.

Let the straight lines AB, CD be each of them parallel to the straight line EF; AB is also parallel to CD.

Let the straight line LH cut AB, CD, EF; and because LH cuts the parallel straight lines AB, EF, the angle LGB is equal (I. 29. part. 2) to the angle LHF. Again, because the straight line LH cuts the parallel straight lines CD, EF, the angle LKD is equal (I. 29, part. 2) to the angle LHF; and it has been shown that the angle LGB is equal to LHF: wherefore also LGB is equal (I. ax. 1) to LKD, the interior and remote angle on the same side of LH: therefore AB is parallel (I. 28, part. 1) to CD. Wherefore two straight lines, &c.

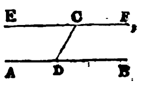


PROP. XXXI. PROB.—To draw a straight line parallel to a given straight line through a given point without it.

Let AB be the given straight line, and C the given point; it is required to draw a straight line through C, parallel to AB.

In AB take any point D, and join CD; at the point C, in the straight line CD, make (I. 23) the angle DCE equal to CDB; and produce the straight line EC to any point F.

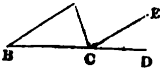
Because (const.) the straight line CD, which meets the two straight lines AB, EF, makes the alternate angles ECD, CDB equal to one another, EF is parallel (I. 27) to AB. Therefore the straight line ECF is drawn through the given point C parallel to the given straight line AB: which was to be done.



PROP. XXXII. THEOR.—If a side of any triangle be produced, (1) the exterior angle is equal to the two interior and remote angles; and (2) the three interior angles of every triangle are together equal to two right angles.

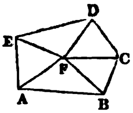
Let ABC be a triangle, and let one of its sides BC be produced to D: (1) the exterior angle ACD is equal to the two interior and remote angles CAB, ABC; and (2) the three interior angles, ABC, BCA, CAB, are together equal to two right angles.

Through the point C draw (I. 31) CE parallel to the straight line AB. Then, because AB is parallel to EC, and AC falls upon them, the alternate angles BAC, ACE are (I. 29, part 1) equal. Again, because AB is parallel to EC, and BD falls upon them, the exterior angle ECD is equal (I. 29, part 2) to the interior and remote angle ABC; but the angle ACE has been shown to be equal to the angle BAC: therefore the whole exterior angle ACD is equal (I. ax. 2) to the two interior and remote angles CAB, ABC. To these equals add the angle ACB, and the angles ACD, ACB are equal (I. ax. 2) to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal (I. 13) to two right angles: therefore also the angles CBA, BAC, ACB are equal to two right angles. Wherefore, if a side, &c.



Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles; and by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is (I. 13, cor. 2), together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal (I. ax. 1) to twice as many right angles as the figure has sides.*



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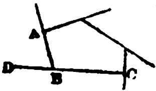
* Another proof of this corollary may be obtained by dividing the figure into triangles by lines drawn from any angle to all the remote angles. Then each of the two extreme triangles has two sides of the polygon for two of its sides, while each of the other triangles has only one side of the figure for one of its sides; and hence the number of triangles is less by two than the number of the sides of the figure. But the interior angles of the figure are evidently equal to all the interior angles of all the triangles, that is, to twice as many right angles as there are triangles, or, twice as many right angles as the figure has sides, wanting the angles of two triangles, that is, four right angles.

Hence, in any equiangular figure, the number of the sides being known, the magnitude of each angle compared with a right angle can be determined. Thus, in a regular pentagon, the amount of all the angles being twice five right angles wanting four, that is six right angles, each angle will be one fifth part of six right angles, *Cor.* 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

Because each interior angle ABC, and the adjacent exterior ABD, are together equal (I. 13) to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal (I. ax. 3) to four right angles.*



Cor. 3. If a triangle have a right angle, the remaining angles are together equal to a right angle; and if one angle of a triangle be equal to the other two, it is a right angle.

Cor. 4. The angles at the base of a right-angled isosceles triangle are each half a right angle.

Cor. 5. If two angles of one triangle be equal to two angles of another, their remaining angles are equal.

Cor. 6. Each angle of an equilateral triangle is one third of two right angles, or two thirds of one right angle.

Cor. 7. Hence, a right angle may be trisected by describing an equilateral triangle on one of the lines containing the right angle.†

PROP. XXXIII. THEOR.—The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts,‡ are also themselves equal and parallel.

Let AB, CD be equal and parallel straight lines, which are joined towards the same parts by the straight lines AC, BD; AC, BD are also equal and parallel.

Join (I. post. 1) BC. Then, because AB is parallel to CD, and

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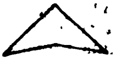
or one right angle and one fifth. In a similar manner it would appear, that in the regular hexagon, each angle is a sixth part of eight right angles, or a right angle and a third; that in the regular heptagon each is a right angle and three sevenths; in the regular octagon, a right angle and a half, &c.

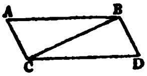
[*] It is to be observed, that if angles be taken in the ordinary meaning, as understood by Euclid, this corollary and the foregoing are not applicable when the figures have *re-entrant* angles, that is, such as open outward. The second corollary will hold, however, if the difference between each re-entrant angle and two right angles be taken from the sum of the other exterior angles: and the former will be applicable, if, instead of the angle which opens externally, the difference between it and four right angles be used. Both corollaries, indeed, will hold without change, if the re-entrant angle be regarded as internal and greater than two right angles; and if, to find the exterior angles, the interior be taken, in the algebraic sense, from two right angles, as in this case, the re-entering angles will give negative or subtractive results.



[†] By this principle also, in connexion with the 9th proposition, we may trisect any angle, which is obtained by the successive bisection of a right angle, such as the half, the fourth, the eighth, of a right angle, and so on.

[‡] That is, the extremities which lie towards the "same parts" or quarter.

BC meets them, the alternate angles ABC, BCD are (I. 29) equal; and because AB is equal (hyp.) to CD, and BC common to the two triangles ABC, DCB, and the angle ABC is equal to the angle BCD: therefore the base AC is equal (I. 4, part 1) to the base BD, and (I. 4, part 3) the angle ACB to CBD. Also, because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal, AC is parallel (I. 27) to BD; and it has been shown to be equal to it. Therefore the straight lines, &c.

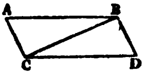


Cor. Hence a quadrilateral which has two sides equal and parallel, is (I. def. 24) a parallelogram.

PROP. XXXIV. THEOR.—The opposite sides and angles of a parallelogram are equal to one another; and (2) the diagonal bisects it.

Let AD[*] be a parallelogram, of which BC is a diagonal; (1) the opposite sides and angles of the figure are equal to one another: and (2) the diagonal bisects it.

Because AB is parallel (I. def. 24) to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29) to one another; and because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are likewise equal: wherefore the two triangles, ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other: therefore (I. 32, cor. 5) the remaining angles BAC, CDB are equal: also the side BC, which is opposite to those angles is common: therefore (I. 26, part 1) their other sides are equal, each to each, viz. AB to CD, and AC to BD. Again, because the angle ABC is equal to BCD, and CBD to ACB, the whole angle ABD is equal (I. ax. 2) to the whole angle ACD; and BAC has been shown to be equal to BDC: therefore the opposite sides and angles of a parallelogram are equal to one another. Also (I. 26, part 3) the triangle ABC is equal to the triangle DCB; that is, the diagonal BC bisects the parallelogram. Therefore the opposite sides, &c.



Cor. 1. If a parallelogram have one right angle, all its angles are right angles. For since (I. 29, part 3) the two angles BAC, ACD, are together equal to two right angles, if one of them be a right angle, the other must also be a right angle; and (by this proposition) the opposite angles are equal. A rectangle (I. def. 26), therefore, has all its angles right angles.

Cor. 2. If two parallelograms have an angle of the one equal to an angle of the other, and the sides containing the equal angles respectively equal, the parallelograms are equal to one another. This will be shown by drawing diagonals subtending the equal angles; as (I. 4) the triangles thus formed are equal; and, by this proposition, the parallelograms are respectively double of those triangles.

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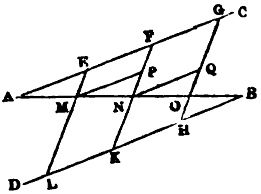
[*] For brevity, a parallelogram may be named by the letters placed at two opposite angles, if no ambiguity arise from so naming it. The same also may be done occasionally with regard to other rectilineal figures.

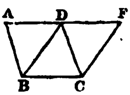
Cor. 3. If two straight lines make an angle, two others parallel to them contain an equal or a supplemental angle. Thus, A and the vertical angle made by producing BA, CA, through A, are each equal to D; while the angle made by BA and the continuation of CA, or by CA and the continuation of BA is equal to the supplement (I. def. 38) of D.

Cor. 4. Hence we can divide a given straight line AB into any proposed number of equal parts. To do this, draw AC making any angle with AB, and draw (I. 31) BD parallel to AC. In AC take any point E, and make (I. 3) EF, FG, BH, HK, KL, each equal to AE; the number of segments on each of the parallels being less by one than the number of the parts into which AB is to be divided. Draw (I. post. 1) the straight lines EL, FK, GH, cutting AB in M, N, O: AM, MN, NO, OB are the parts required. Draw MP parallel to AC; then (I. 33) EL, FK, GH are parallel; and AE and MP are equal, each of them being equal (const. and I. 34) to EF. But (I. 29) the angles EAM, AME are respectively equal to PMN, MNP; and therefore (I. 26) AM is equal to MN. In the same manner, BO may be proved to be equal to ON: and by drawing NQ parallel to AC, MN may be proved in a similar manner to be equal to NO.*



PROP. XXXV. THEOR.—Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF, (figs. 2 and 3) be upon the same base BC, and between the same parallels AF, BC; these parallelograms are equal to one another.



If the sides AD, DF, of the parallelograms AC, BF (fig. 1) opposite to the base BC, be terminated in the same point D, each of the parallelograms is double (I. 34) of the triangle BDC; and they are therefore equal (I. ax. 6) to one another.

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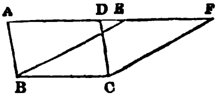
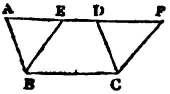
* The construction might be effected as in the 9th of the sixth book without the parallel BD. The method here given, however, is easier in practice, from the facility with which the parallels EL, FK, &c. are drawn. This problem might be considered unnecessary here, as it is the same in substance as the 9th of the sixth book. Besides the preferable construction, however, which is here given, the problem is so useful, that the student should be early acquainted with it.

But if the sides AD, EF (figs. 2 and 3) opposite to the base BC, be not terminated in the same point; then, because AC is a parallelogram, DC is equal (I. 34) to AB. Also (I. 29, part 2) because AB is parallel to DC, and AF falls upon them, the angles FDC, EAB are equal; and because BE is parallel to CF, and AF falls upon them, the angles DFC, AEB are likewise equal: wherefore (I. 26, part 3) the triangles FDC, EAB are equal to one another. Take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB, and (I. ax. 3) the remainders are equal; that is, the parallelogram AC is equal to the parallelogram BF.* Therefore parallelograms, &c.

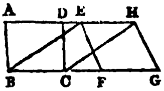




Prop. XXXVI. Theor.—Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let AC, EG be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG; the parallelogram AC is equal to EG.

Join BE, CH; and because BC is equal (hyp.) to FG, and (I. 34) FG to EH; BC is equal (I. ax. 1) to EH; and (hyp.) they are parallel, and they are joined towards the same parts by the straight lines BE, CH: therefore (I. 33) BE, CH are both equal and parallel, and (I. def. 24) EBCH is a parallelogram; and it is equal (I. 35) to AC, because they are upon the same base BC, and between the same parallels BC, AH. For the like reason, the parallelogram EG is equal to the same parallelogram EBCH; because they are on the same base EH, and between the same parallels EH, BG. Therefore also the parallelogram AC is equal (I. ax. 1) to EG. Wherefore parallelograms, &c.



Prop. XXXVII. Theor.†—Triangles upon the same base, and between the same parallels, are equal to one another.

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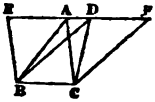
* It will appear, from this proposition, that the perimeters of two equal parallelograms on the same base may differ in any degree whatever. It follows also, from the 19th proposition, that, of all equal parallelograms on the same base, the rectangle has the least perimeter.

It may also be remarked, that Legendre and some other writers call only such figures *equal* as exactly coincide when applied to one another; while those which, as in this proposition, "without coinciding, contain the same space," they call *equivalent*. This seems to be an unnecessary refinement.

† This proposition and the following afford instances in which triangles, though not as in propositions 4, 8, and 26, in *every respect* equal, are equal in *magnitude*.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC; ABC is equal to DBC.

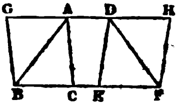
Produce AD both ways to E and F, and through B draw (I. 31) BE parallel to CA, and through C draw CF parallel to BD.* Therefore (I. def. 24) each of the figures EC, BF is a parallelogram; and (I. 35) they are equal to one another, because they are upon the same base BC, and between the same parallels BC, EF. But the triangle ABC is the half of the parallelogram EC, because (I. 34) the diagonal AB bisects it; and the triangle DBC is the half of the parallelogram BF, because the diagonal DC bisects it: but (I. ax. 7) the halves of equals are equal; therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles, &c.



PROP. XXXVIII. THEOR.—Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases, BC, EF, and between the same parallels BF, AD: ABC is equal to DEF.

Produce AD both ways to G and H, and (I. 31) through B draw BG parallel to CA, and through F draw FH parallel to ED. Then (I. def. 24) each of the figures GC, EH is a parallelogram; and (I. 36) they are equal to one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH; and (I. 34) the triangle ABC is half of the parallelogram GC; and the triangle DEF of the parallelogram EH. But (I. ax. 7) the halves of equals are equal: therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles, &c.



Cor. 1. From this proposition and from the 36th, it is plain, that triangles and parallelograms between the same parallels, but upon unequal bases, are unequal; that which has the greater base being greater than the other.

Cor. 2. Hence a straight line drawn from the vertex of a triangle to the point of bisection of the base, bisects the triangle; and if two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles (I. def. 38) supplemental, the triangles are equal.

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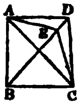
* It follows from the first corollary to the 29th proposition, that AD, when produced, will meet BE and CF: and in the next proposition, AD produced will meet the corresponding lines on the same principle. Euclid often assumes without proof, that certain lines will meet. The omission of the proof, however, is of no consequence, as it can occasion no doubt or difficulty.

PROP. XXXIX. THEOR.[*]—**If equal triangles, not coinciding, stand on the same base, and on the same side of it, the straight line joining their vertices is parallel to the base.**

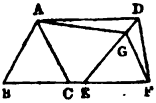
Let the equal triangles ABC, DBC be on the same base BC and on the same side of it; join AD: AD is parallel to BC.



For, if it be not, through A draw (I. 31) AE parallel to BC, and join EC. Then (I. 37) the triangles ABC, EBC are equal, because they are on the same base BC, and between the same parallels BC, AE. But the triangle ABC is equal (hyp.) to the triangle DBC: therefore also the triangle EBC is equal (I. ax. 1) to the triangle DBC, the less to the greater, which is impossible. Therefore AE is not parallel to BC; and in the same manner it can be demonstrated, that no other line drawn through A, except AD, is parallel to BC: AD is therefore parallel to it. Wherefore, if equal triangles, &c.

PROP. XL. THEOR.—If equal triangles, not having a common vertex, stand upon equal bases in the same straight line, and on the same side of it, the straight line joining their vertices is parallel to the line on which they stand.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and on the same side of it; join AD: AD is parallel to BF.



For, if it be not, through A draw (I. 31) AG parallel to BF, and join GF. Then (I. 38) the triangles ABC, GEF are equal, because they are on equal bases BC, EF, and between the same parallels BF, AG. But the triangle ABC is equal (hyp.) to the triangle DEF: therefore also the triangle GEF is equal to the triangle DEF, the less to the greater, which is impossible. Therefore AG is not parallel to BF; and in the same manner it can be demonstrated, that through A no other straight line can be drawn parallel to BF, but AD: AD is therefore parallel to BF. Wherefore, if equal triangles, &c.

PROP. XLI. THEOR.—If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE; the parallelogram is double of the triangle.

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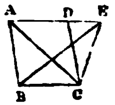
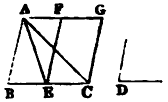
[*] In this proposition and the following, we have two other principles by which lines are proved to be parallel, in addition to those contained in propositions 27, 28, 30, and 33.



Join AC. Then (I. 37) the triangles ABC, EBC are equal, because they are upon the same base BC, and between the same parallels BC, AE. But (I. 34) the parallelogram ABCD is double of the triangle ABC, because the diagonal AC bisects it: wherefore ABCD is also double of the triangle EBC. Therefore, if a parallelogram, &c.

PROP. XLII. PROB.—**To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.**

Let ABC be the given triangle, and D the given angle: it is required to describe a parallelogram equal to ABC, and having one of its angles equal to D.

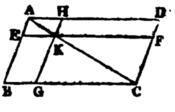


Bisect (I. 10) BC in E, join AE, and (I. 23) at the point E, in the straight line BC, make the angle CEF equal to D; through A draw (I. 31) AG parallel to BC, and through C draw CG parallel to EF; FECG is the parallelogram required.

For, since (const.) BE is equal to EC, the triangle ABE is equal (I. 38, cor. 2) to the triangle AEC: therefore the triangle ABC is double of the triangle AEC. The parallelogram EG is likewise double (I. 41) of the triangle AEC, because they are upon the same base, and between the same parallels. Therefore the parallelogram EG is equal (ax. 6) to the triangle ABC, and it has (const.) one of its angles CEF equal to D: wherefore there has been described a parallelogram EG equal to a given triangle ABC, and having one of its angles CEF equal to the given angle D: which was to be done.

PROP. XLIII. THEOR.—If through any point in either diagonal of a parallelogram, straight lines be drawn parallel to the sides; of the four parallelograms thus formed, those through which the diagonal does not pass, and which are called the *complements* of the other two, are equal.

Let BD be a parallelogram, of which AC is one of the diagonals: then, if through K a point in AC, the two straight lines EF and HG be drawn parallel to the sides, the parallelograms BK and KD are equal.*



Because BD is a parallelogram, and AC its diagonal, the triangle ABC is equal (I. 34) to the triangle ADC: and, because EH is a parallelogram, the diagonal of which is AK, the triangles AEK, AHK are equal. For the same reason, the triangles KGC, KFC are equal. Then, because the triangle AEK is equal to AHK, and KGC to KFC, the triangles AEK, KGC are together equal (I. ax. 2) to the triangles AHK, KFC taken together. But the whole triangle ABC is equal to the whole ADC: therefore (I. ax. 3) the remaining complements BK, KD are equal.* Wherefore, if through any point, &c.

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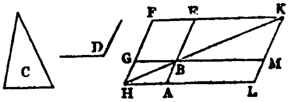
* EH and GF are generally called the *parallelograms about the diagonal*, while BK and KD are called the *complements of* EH and GF, because, with these, they *complete* the entire parallelogram BD.

PROP. XLIV. PROB.—To a given straight line to apply† a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given angle.

Let AB be a given straight line, C a given triangle, and D a given angle. It is required to apply to AB a parallelogram equal to C, and having an angle equal to D.

Make (I. 42) the parallelogram BF equal to the triangle C, having the angle EBG equal to the angle D, and so that BE may be in the same straight line with AB,‡ and produce FG to H. Through A draw (I. 31) AH parallel to BG or EF, and join HB. Then, because (I. def. 24) GB and FE are parallel, and that HB intersects one of them GB, it will also (I. 29, cor. 1) intersect the other FE, if sufficiently produced: let K be the point of intersection, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M. Then FL is a parallelogram, of which the diagonal is HK, and AG, ME are the parallelograms about HK; and LB, BF are the complements: therefore LB is equal (I. 43) to BF: but BF is equal to the triangle C; wherefore LB is equal (I. ax. 1) to C; and because the angle GBE is equal (I. 15) to ABM, and likewise to D, the angle ABM is equal (I. ax. 1) to D. Therefore the parallelogram LB is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to D: which was to be done.



PROP. XLV. PROB.—To describe a parallelogram equal

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* By adding first EH, and then GF, to the complements, we have the parallelograms ED and BF respectively equal to BH and GD: and hence it follows, that if lines, EF, HG, be drawn parallel to the sides, through any point K in the diagonal, the parts into which one of them divides the parallelogram, are respectively equal to those into which it is divided by the other.

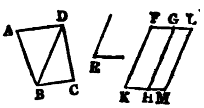
† To *apply* a parallelogram to a given straight line, is to describe a parallelogram having that line for one of its sides.

‡ In strictness, a parallelogram should be constructed (I. 42) equal to the triangle C, and having an angle equal to D. Then, AB being produced, the parallelogram BF should be constructed, by propositions 23, 3, and 31, having the angle EBG equal to D, and the sides EB, BG respectively equal to two adjacent sides of the parallelogram previously constructed: and this parallelogram being equal (I. ax. 1; and I. 34, cor. 3) to the triangle C, the rest of the construction will proceed as in the text.

to a given rectilineal figure, and having an angle equal to a given angle.

Let ABCD be a given rectilineal figure, and E a given angle. It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Join DB, and describe (I. 42) the parallelogram FH equal to the triangle ADB, and having the angle K equal to E. Produce KH to M, and to the straight line GH apply (I. 44) the parallelogram GM equal to the triangle DBC, and having GHM for one of its angles: and because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal: add to each of these the angle HGL: therefore the angles MHG, HGL are equal to HGF, HGL: but MHG, HGL are equal (I. 29) to two right angles: wherefore also HGF, HGL are equal to two right angles, and FG is therefore (I. 14) in the same straight line with GL. Because also KF is parallel to HG, and HG to ML; KF is parallel (I. 30) to ML: and KM, FL are parallels: wherefore (I. def. 24) KL is a parallelogram. And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM; the whole figure ABCD is equal to the whole parallelogram KL. Therefore the parallelogram KL has been described equal to the given rectilineal figure ABCD, and having the angle K equal to the given angle E: which was to be done.

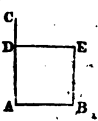


Cor. From this it is manifest how to a given straight line a parallelogram may be applied, which shall have an angle equal to a given angle, and shall be equal to a given rectilineal figure; viz. by applying (I. 44) to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle. The rest of the construction will be the same as in the proposition.

PROP. XLVI. PROB.—To describe a square upon a given straight line.

Let AB be a given straight line; it is required to describe a square upon it.

From the point A draw (I. 11) AC at right angles to AB, and make AD equal to AB:* through the point D draw (I. 31) DE parallel to AB, and through B draw BE parallel to AD. Then (I. def. 26, and const.) the figure AE is a rectangle; and since AD, AB are equal, AE is (I. def. 27) a square; and being described on the given line AB, it is the square required.



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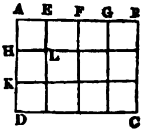
* The part of the construction that follows is effected more simply in practice, by describing arcs from B and D as centres, and with radii equal to AB, as these will intersect each other in E. In the construction given in the text, it should be

Cor. 1. Hence, (const. and I. 34) a square has all its sides equal: and (I. 34, cor. 1) all its angles are right angles.

Cor. 2. Hence the squares described on equal straight lines (I. 34, cor. 2) are equal.

Cor. 3. If two squares be equal, their sides are equal. For, if the sides be not equal, from one of them cut off a part equal to the other, and on that part describe a square. This square (by the preceding corollary, and I. ax. 1) will be equal to the square of which it is a part, which is absurd.



Cor. 4. If AB and AD, two adjacent sides of a rectangle BD, be divided into parts AE, EF, AH, &c. which are all equal; straight lines drawn through E, F, H, &c. the points of section, parallel to the sides, divide the rectangle into squares HE, FL, KL, &c. which are all equal, and the number of which is equal to the product of the number of parts in AB, one of the sides, multiplied by the number of parts in AD, the other. For (const.) these figures are all parallelograms; and (I. def. 27, and const.) HE is a square. FL is also a square, its sides EF and EL being each equal to AE, and the angle LEF being (I. 29, part 2) equal to A, and therefore a right angle: and in a similar manner all the others might be proved to be squares; and (I. 34, cor. 2) they are all equal. Of these squares also there are evidently as many *columns* as there are parts in AB; while in each column there are as many squares as there are parts in AD. The number of such squares contained in a figure is called, in the language of mensuration, the *area* of that figure.*

Cor. 5. Hence, since any parallelogram is equal (I. 35) to a rectangle on the same base and between the same parallels, it follows, that *the area of any parallelogram is equal to the product of its base and its perpendicular height:* and *the area of a square is computed by multiplying a side by itself.*

Cor. 6. Hence, also, (I. 41) *the area of a triangle is computed by multiplying any of its sides by the perpendicular drawn to that side from the opposite angle, and taking half the product: and the area of a trapezium is found by multiplying either diagonal by the sum*† *of the perpendiculars drawn to it from the angles which it subtends, and taking half the product.*

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strictness be proved that DE and BE will meet. This is easily done by means of the 12th axiom, if BD be joined. It is plain that there might be another square described on AB, on the other side.

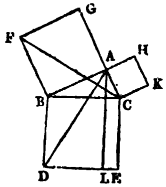
* Thus, if the length of a rectangle be 4 feet, and its breadth three feet, it will contain 12 squares, the side of each of which is one foot; or, as it is briefly expressed, its area is 12 *square feet:* and a rectangle 15 inches long, and 10 inches broad, contains 150 *square inches;* that is, 150 squares, each having its side an inch.

† When, in consequence of one of the angles being re-entrant, the perpendiculars lie on the same side of the diagonal, the *difference* of the perpendiculars must evidently be used, instead of their sum.

Cor. 7. Every polygon may be divided into triangles or trapeziums by drawing diagonals; and, therefore, the area of any polygon whatever can be computed by finding the areas of those component figures, by the last corollary, and adding them together.*

PROP. XLVII. THEOR.—In any right-angled triangle, the square which is described upon the hypotenuse, that is, *the side subtending the right angle*, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC; the square described upon the hypotenuse BC is equal to the squares described upon BA and AC.



On BC describe (I. 46) the square BE, and on BA, AC, the squares GB, HC; through A draw (I. 31) AL parallel to BD, or CE, and join AD and FC. Then, because each of the angles BAC, BAG is a right angle, the two straight lines AC, AG upon the opposite sides of AB, make with it, at the point A, the adjacent angles equal to two right angles; therefore CA is in the same straight line (I. 14) with AG. For the same reason, AB and AH are in the same straight line. Again, because the angle DBC is equal to FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is equal (I. ax. 2) to the whole FBC: and because the two sides AB, BD are equal (const.) to the two FB, BC, each to each, and the angle DBA equal to FBC; therefore the triangle ABD is equal (I. 4, part 2) to the triangle FBC. Now (I. 41) the parallelogram BL is double of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC. But the doubles of equals are equal (I. ax. 6) to one another: therefore the parallelogram BL is equal to the square GB. In the same manner, by joining AE, BK, it would be demonstrated that the parallelogram CL is equal to the square HC. Therefore the whole square BE is equal (I. ax. 2) to the two squares GB, HC; and the square BE is described upon the hypotenuse BC, and the squares GB, HC upon

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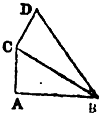
* This corollary and the two foregoing contain the elementary principles of the mensuration of rectilineal figures, and they form a connexion between arithmetic or algebra, and geometry. They also explain the origin of the expressions, "the square of a number," "the rectangle of two numbers," and "the product of two lines." The last of these is a convenient abbreviation for the product of the number of equal parts in one line by those in another. The second is not only an improper, but an unnecessary substitute for "the product of two numbers." Instead of the first it would be better to employ "the second power of a number;" but the expression is in so general use that it is not likely to be laid aside.

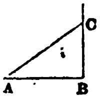
BA, AC: wherefore the square upon BC is equal to the squares upon BA and AC.[*] Therefore, in any right-angled triangle, &c.

Cor. 1. Hence, if the lengths of any two sides of a right-angled triangle be given in numbers, the remaining side can be found.[†]

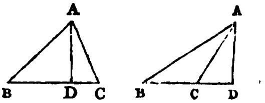
Cor. 2. Hence, also, we may find a square equal to the sum of two or more given squares. Thus, let AB be the side of one square, and AC perpendicular to it, the side of another, and join BC: the square of BC is equal to the sum of the squares of AB and AC. In like manner, if CD be drawn perpendicular to BC, and equal to the side of a third square, and if BD be joined, the square of BD is equal to the squares of BC, CD, or to those of BA, AC, CD: and thus a square may be found equal to the sum of any number of given squares.



Cor. 3. A square may also be found equal to the difference of two given squares. Thus, to AB the side of the less square draw the perpendicular BC; and from A as centre, with the side of the greater as radius, describe an arc cutting the perpendicular in C: the square of BC will be equal to the difference of the given squares.



Cor. 4. If a perpendicular AD be drawn from one of the angles of the triangle ABC, cutting the opposite side in D, the difference of the squares of the sides AB, AC is equal to the difference of the squares of the segments BD, CD intercepted between the extremities of the base and the point in which the perpendicular cuts it. For the square of AB is equal to the squares of BD and DA, and the square of AC to the squares of CD and DA; and by taking the latter equals from the former, there remains the difference of the squares of AB and AC equal to the difference of the squares of the segments BD and CD.[‡]



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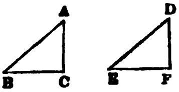
[*] In the proof here given, which is as simple and easy as any other that has been proposed instead of it, the squares are all described externally with respect to the triangle. This, however, is not essential, as the squares may be described either all on the other sides of the lines AB, AC, BC, or some on one side and some on the other. The proof is nearly alike in all these modes, the chief difference consisting in using, in some instances, instead of the 4th proposition, the 2d corollary to the 38th. This proposition and the fourth of the sixth book are the most valuable propositions in the whole of elementary geometry.

[†] Thus, if AB be 15 inches, and AC 8 inches, the areas of the squares BG and CH are (I. 46, cor. 5) 225 and 64 square inches. The sum of these, 289, is by this proposition, the area of BE; the square root of which is 17 inches, the length of BC. In like manner, if BC be 13 perches, and AC 5 perches, the areas of BE and CH are 169 and 25 square perches; the difference of which, 144, is, by this proposition, the area of BG; whence, by extracting the square root, AB is found to be 12 perches.

[‡] In this proof the obvious principle is employed, that the difference of two magnitudes is the same as the difference obtained after adding to each the same third magnitude. Thus, the difference of the squares of BD and DC is the same as the difference between the sum of the squares of BD and DA, and of DC and DA.

Cor. 5. If two right-angled triangles ABC, DEF have their hypotenuses AB, DE equal, and likewise the sides BC, EF equal; the remaining sides AC, DF are also equal; and the angle A is equal to D, and B to E. For, since the hypotenuses are equal, the squares of BC and CA are equal to the squares of EF and FD; of which also the squares of BC and EF are equal; and therefore (I. ax. 3) the squares of AC and DF, and consequently (I. 46, cor. 3) AC and DF themselves are equal. Also, since the sides of the triangles are respectively equal, their angles (I. 8) are likewise equal, each to each.



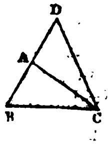
The converse of this, that if the legs* of two right-angled triangles be equal, each to each, their hypotenuses are also equal, may be proved in a similar manner, or by means of the 4th proposition of this book.

Cor. 6. If the hypotenuses AB, DE be equal, but BC, a leg of one of them, be greater than EF, a leg of the other, the remaining leg AC of the first triangle is less than DF, the remaining leg of the other. For, as was seen in the preceding corollary, the squares of BC, CA are equal to the squares of EF, FD; from which, if the unequal squares of BC, EF be taken, the remaining square of AC is less than the remaining square of DF; and, therefore, AC is manifestly less than DF.

PROP. XLVIII. THEOR.—If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides, the angle contained by those two sides is a right angle.

If the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides, BA, AC, the angle BAC is a right angle.

From the point A draw (I. 11) AD perpendicular to AC, and equal to BA; and join CD. Then, because DA is equal to AB, the squares of DA and AB are (I. 46, cor. 2) equal. To each of these add the square of AC; therefore the squares of DA, AC are equal to the squares of BA, AC. But (I. 47) the square of DC is equal to the squares of DA, AC, because DAC is a right angle; and the square of BC is equal (hyp.) to the squares of BA, AC; therefore the square of DC is equal (I. ax. 1) to the square of BC; and therefore also the side DC is equal (I. 46, cor. 3) to the side BC. And because the side DA is equal to AB, and AC common to the two triangles DAC, BAC, and the base DC equal to the base BC: therefore (I. 8) the angles DAC, BAC are equal: but (const.) DAC is a right angle; wherefore also BAC is a right angle. Therefore, if the square, &c.



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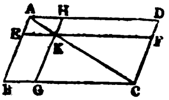
* The sides containing the right angle are often called the *legs* of the triangle.

BOOK II.*

DEFINITIONS.

1. A RECTANGLE is said to be *contained* by the two straight lines which are about any of the right angles.†

2. In every parallelogram, the figure which is composed of either of the parallelograms about a diagonal, and of the two complements, is called a *Gnomon*.‡ "Thus the "parallelogram HE together with the complements BK, KD, is a gnomon, which is "more briefly expressed by the letters BHF, "or GED, which are at the opposite angles "of the parallelograms that make up the "gnomon.



"For the sake of brevity, the rectangle "contained by AB and CD is often expressed simply by AB.CD, "a point being placed between the letters denoting the sides of the "rectangle: and the square of a line AB is often written simply "AB^2.

"In Algebra, the sign +, called *Plus* (*more by*) placed between "the names of two magnitudes, is used to denote that these mag"nitudes are added together; and the sign —, called *Minus* (*less* "*by*) placed between them, to signify that the latter is taken from "the former. The sign =, which is read *equal to*, signifies that "the quantities between which it stands, are equal to one another. "These signs will be occasionally used for brevity, in what follows."

PROPOSITIONS.

PROP. I. THEOR.—If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by A and BC is equal to the rectangles contained by A and BD, A and DE, and A and EC.

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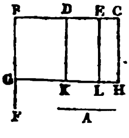
* This book treats of right-angled parallelograms, and in particular of squares.

† That is, by any two adjacent sides, or by its length and its breadth. The rectangle contained by two straight lines is sometimes called simply the rectangle *under* those lines, or the rectangle *of* them.

‡ A gnomon might be more simply defined to be the part of a parallelogram which remains, when either of the parallelograms about one of the diagonals is taken away. The term, however, might be altogether disused, as it is seldom employed either by Euclid or any other writer, and scarcely any circumlocution would arise from laying it aside.

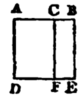
From B draw (I. 11) BF at right angles to BC, and (I. 3) make BG equal to A: through G draw (I. 31) GH parallel to BC; and through D, E, C, draw DK, EL, CH parallel to BG. Then BH, BK, DL, and EH are evidently rectangles; and BH is equal (I. ax. 10) to BK, DL, EH. But BH is contained by A and BC, for (II. def. 1) it is contained by GB and BC, and GB is equal (const.) to A; and BK is contained by A and BD, for it is contained by GB and BD, of which GB is equal to A. Also DL is contained by A and DE, because DK, that is, (I. 34) BG, is equal to A: and in like manner it is shown that EH is contained by A and EC. Therefore the rectangle contained by A and BC is equal to the several rectangles contained by A and BD, by A and DE, and by A and EC.* Wherefore, if there be two straight lines, &c.



PROP. II. THEOR.—If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in the point C; the rectangles AB.AC, and AB.BC are equal to the square of AB.

Upon AB describe (I. 46) the square AE, and through C draw (I. 31) CF parallel to AD or BE. Then AE is equal (I. ax. 10) to the rectangles AF and CE. But AE is the square of AB, and AF is the rectangle contained by BA, AC: for (II. def. 1) it is contained by DA, AC, of which DA is equal to AB: and CE is contained by AB, BC, for BE is equal to AB: therefore the rectangles under AB, AC, and AB, BC are equal to the square of AB.† If, therefore, &c.



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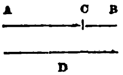
* This demonstration depends on the principle that the whole is equal to all its parts, or rather that it is identical with them: for BH is equal to BK, DL, and EH; but (II. def. 1) BH is equal to the rectangle under A and BC, and BK, DL, and EH are equal to those contained by BD and A, DE and A, and EC and A.

† This proposition may also be demonstrated in the following manner:—

Take a straight line D equal to AB. Then (II. 1) the rectangles AC.D and BC.D are together equal to AB.D. But since D is equal to AB, the rectangle AB.D is equal (I. def. 27) to the square of AB, and the rectangles AC.D and BC.D are respectively equal to AC.AB and BC.AB: wherefore the rectangles AC.AB and BC.AB are together equal to the square of AB.

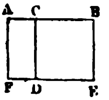
In a manner similar to this, several of the following propositions may be demonstrated. Such proofs, though perhaps not so easily understood at first by the learner, are shorter than those given by Euclid; and they have the advantage of being derived from those preceding them, instead of being established by continual appeals to original principles.



PROP. III. THEOR.—**If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the square of that part, together with the rectangle contained by the two parts.**

Let the straight line AB be divided into two parts in the point C; the rectangle AB.BC is equal to the square of BC, together with the rectangle AC.CB.

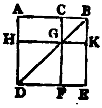
Upon BC describe (I. 46) the square CE; produce ED to F; and through A draw (I. 31) AF parallel to CD or BE. Then the rectangle AE is equal (I. ax. 10) to the rectangles CE, AD. But AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; also DB is the square of BC. Therefore the rectangle AB.BC is equal to the square of BC, together with the rectangle AC.CB.* If, therefore, &c.



PROP. IV. THEOR.—**If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice their rectangle.**†

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC and CB, together with twice the rectangle under AC and CB.

On AB describe (I. 46) the square AE, and join BD; through C draw (I. 31) CGF parallel to AD or BE; and through G draw HK parallel to AB or DE. Then, because CF is parallel to AD, and BD falls upon them, the exterior angle CGB is equal (I. 29) to the interior and remote angle ADB; but ADB is equal (I. 5) to ABD, because BA and AD are equal, being sides of a square: wherefore (I. ax. 1) the angle CGB is equal to GBC: and therefore the side BC is equal (I. 6) to the side CG. But (const.) the figure CK is a parallelogram; and since CBK is a right angle, and BC equal to CG, CK (I. def. 26 and 27) is a square, and it is upon the side CB. For the like reason HF also is a square, and it is upon the side HG, which is equal (I. 34) to AC: therefore HF, CK are the squares of AC, CB. And because (I. 43) the complements AG, GE are equal, and that AG is the rectangle contained by AC, CB, for CG has been proved to be equal to CB: therefore



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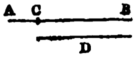
* Otherwise:—Take a line D equal to CB. Then (II. 1) the rectangle AB.D is equal to the rectangles BC.D and AC.D; that is (const. and I. def. 27), the rectangle AB.BC is equal to the square of BC together with the rectangle AC.CB.



† The enunciation may also be thus expressed:—*The square of the sum of two straight lines is equal to the sum of their squares, together with twice their rectangle.*

GE is also equal to the rectangle AC.CB; wherefore AG, GE are equal to twice the rectangle AC.CB. The four figures, therefore, HF, CK, AG, GE are equal to the squares of AC, CB, and twice the rectangle AC.CB. But HF, CK, AG, GE make up the whole figure AE, which is the square of AB: therefore the square of AB is equal to the squares of AC and CB, and twice the rectangle AC.CB.[*] Wherefore, &c.

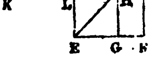
Cor. 1. It follows from this demonstration, that the parallelograms about the diagonal of a square are likewise squares.

Cor. 2. Hence the square of a straight line is equal to four times the square of its half: for the straight line being bisected, the rectangle of the parts is equal to the square of one of them.

Prop. V. Theor.—If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be bisected in C, and divided unequally in D: the rectangle AD.DB, together with the square of CD, is equal to the square of CB.

On CB describe (I. 46) the square CF, join BE, and through D draw (I. 31) DHG parallel to CE or BF; also through H draw KLM parallel to CB or EF; and through A draw AK parallel to CL or BM. Then (I. 36) AL and CM are equal, because AC is equal to CB; and (I. 43) the complements CH and HF are equal. Therefore (I. ax. 2) AL and CH together are equal to CM and HF together; that is, AH is equal to the gnomon CMG. To each of these add LG, and (I. ax. 2) the gnomon CMG, together with LG, is equal to AH together with LG. But the gnomon CMG, and LG make up the figure CEFB, which is the square of CB: also AH is the rectangle under AD and DB, because DB is equal (II. 4. cor. 1) to DH; and LG is the square of CD. Therefore the rectangle AD.DB and the square of CD are equal to the square of CB.[†] Wherefore, if a straight line, &c.



Cor. 1. Hence the difference of the squares of CB and CD is equal to the rectangle under AD and DB. But since AC is equal to CB, AD is equal to the sum of CB and CD, and DB is the difference of these lines. Hence *the difference of the squares of two straight lines is equal to the rectangle under their sum and difference.*

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[*] Otherwise:—$AB^2 = AB.AC + AB.CB$ (II. 2). But (II. 3) $AB.AC = AC^2 + AC.CB$, and $AB.CB = CB^2 + AC.BC$. Hence (I. ax. 2) $AB.AC + AB.BC$, or $AB^2 = AC^2 + CB^2 + 2AC.CB$.

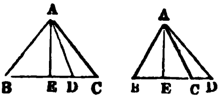
[†] Otherwise:—Since, as is easily seen from the proof in the text, DF is equal to AL, take these separately from the entire figure, and there remain AH and LG equal to the square CF, as before. The proof may also be as follows:—

$AD.DB = CD.DB + AC.DB$ (II. 1) or $AD.DB = CD.DB + CB.DB$, because $CB = AC$. Hence (II. 3) $AD.DB = CD.DB + CD.DB + DB^2 = 2CD.DB + DB^2$. To each of these add CD^2: then $AD.DB + CD^2 = 2CD.DB + DB^2 + CD^2$, or (II. 4) $AD.DB + CD^2 = CB^2$.

Cor. 2. Since the square of CB, or which is the same, the rectangle AC.CB, is greater than the rectangle AD.DB by the square of CD, it follows, that to divide a straight line into two parts, the rectangle of which may be the greatest possible, or, as is termed, a *maximum*, the line is to be bisected.

Cor. 3. Hence also the sum of the squares of the two parts into which a straight line is divided, is the least possible, or is, as it is termed, a *minimum*, when the line is bisected. For (II. 4) the square of the line is equal to the squares of the parts and twice their rectangle; and therefore the greater the rectangle is, the less are the squares of the parts: but, by the foregoing corollary, the rectangle is a maximum when the line is bisected.

Cor. 4. Since (I. 47, cor. 4) the difference of the squares of the sides of a triangle is equal to the difference of the squares of the segments of the base, it follows, from the first corollary above, that the rectangle under the sum and difference of the sides of a triangle, is equal to the rectangle under the sum and difference of the segments, intercepted between the extremities of the base and the point in which the perpendicular cuts the base, or the base produced.



Cor. 5. Hence also, if a straight line AD be drawn from the vertex A of an isosceles triangle ABC, to any point D in the base, or its continuation, the difference of the squares of that line and either of the equal sides is equal to the rectangle under BD and DC, the segments intercepted between the extremities of the base and the point D. To prove this, bisect BC in E, and join AE: then (I. 8) AE is perpendicular to BC. But, by the last corollary, the difference of the squares of AB and AD is equal to the rectangle under the sum and difference of BE and ED: that is, to the rectangle under BD and DC, since BE is equal to EC.

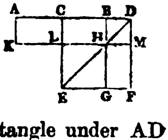
Cor. 6. Since (I. 47) the square of one of the legs of a right-angled triangle is equal to the difference of the squares of the hypotenuse and the other leg, it follows (II. 5, cor. 1) that the square of one leg of a right-angled triangle is equal to the rectangle under the sum and difference of the hypotenuse and the other.

PROP. VI. THEOR.—If a straight line be bisected, and be produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

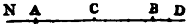
Let the straight line AB be bisected in C, and produced to D;

the rectangle AD.DB, and the square of CB, are equal to the square of CD.

Upon CD describe (I. 46) the square CF, and join DE; through B draw (I. 31) BHG parallel to CE or DF; through H draw KLM parallel to AD or EF, and through A draw AK parallel to CL or DM. Then, because AC is equal to CB, the rectangle AL is equal (I. 36) to CH; but (I. 43) the complements CH, HF are equal: therefore also AL is equal to HF. To each of these add CM and LG; therefore AM and LG are equal to the whole square CEFD. But AM is the rectangle under AD and DB, because (II. 4, cor. 1) DB is equal to DM: also LG is the square of CB, and CEFD the square of CD. Therefore the rectangle AD.DB and the square of CB are equal to the square of CD.* Wherefore, if a straight line, &c.



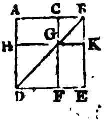
OTHERWISE:—Produce CA to N, and make CN equal to CD. To these add the equals CB and CA; therefore NB is equal (I. ax. 2) to AD. But (II. 5) the rectangle NB.BD, or AD.BD, together with the square of CB, is equal to the square of CD: which was to be proved.



PROP. VII. THEOR.—If a straight line be divided into any two parts, the squares of the whole line and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC are equal to twice the rectangle AB.BC, together with the square of AC.

Upon AB describe (I. 46) the square AE, and construct the figure as in the preceding propositions. Then, because (I. 43) the complements AG, GE are equal, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE are together double of AK. But AK, CE are the gnomon AKF, together with the square CK: therefore the gnomon AKF and the square CK are double of AK, or double of the rectangle AB.BC, because BC is equal (II. 4, cor. 1) to BK. To each of these equals add HF, which is equal (II. 4, cor. 1) to the square of AC; therefore the gnomon AKF, and the squares CK, HF are equal to twice the rectangle AB.BC and the square of AC. But the gnomon AKF, and the squares CK, HF make up the whole figure AE, together with CK; and these are the squares of AB and BC: therefore the squares of



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* After it has been proved, as above, that HF is equal to AL, the rest of the proof will be simplified by taking these separately from the whole figure, as there will remain the rectangle AM and the square LG, together equal to the square CF.

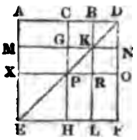
AB and BC are equal to twice the rectangle AB.BC, together with the square of AC:* wherefore, if a straight line, &c.

Cor. 1. Since AC is the difference of AB and BC, it follows that the square of the difference of two straight lines is equal to the sum of their squares, wanting twice their rectangle.

Cor. 2. Since (II. 4) the square of the sum of two lines exceeds the sum of their squares, by twice their rectangle, and, since, by the foregoing corollary, the square of their difference is less than the sum of their squares, by twice their rectangle, it follows that the square of the sum of two lines, is equal to the square of their difference, together with four times their rectangle.

PROP. VIII. THEOR.—If a straight line be divided into any two parts, four times the rectangle contained by the whole and one of the parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole and the former part.

Let the straight line AB be divided into any two parts in C; four times the rectangle AB.BC, and the square of AC, are equal to the square of the line made up of AB and BC together.



Produce AB to D, so that BD may be equal to CB, and upon AD describe the square AF: join DE, and draw BKL, CPH parallel to AE or DF, and NKGM, XPRO parallel to AD or EF. Because CB is equal (const.) to BD, and that CB is equal (I. 34) to GK, and BD to KN: therefore GK is equal to KN. For the same reason PR is equal to RO: and because CB is equal to BD, and GK to KN, the rectangle CK is equal (I. 36) to BN, and GR to RN. But CK is equal (I. 43) to RN; therefore also BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another, and so are quadruple of one of them CK. Again, because CB is equal to BD, and that BD is equal (II. 4, cor. 1) to BK, that is, to CG; and CB equal to GK, that is, to GP; therefore CG is equal to GP; and because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF. But MP is equal (I. 43) to PL, because they are the complements of the parallelogram ML; wherefore AG is equal also to RF. Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so are quadruple of one of them AG. It was also demonstrated, that the four CK, BN, GR, and RN are quadruple of CK: therefore the eight rectangles which make up the gnomon AOH, are quadruple of AK:† and because AK is contained by AB, BK, or AB, BC, four times the rectangle AB.BC, is quadruple of AK. But the gnomon AOH was demonstrated to be quadruple of AK; therefore four times the rectangle AB.BC is equal to AOH. To each of these add XH, which is equal (II. 4, cor. 1) to the square of AC; then four times the rectangle AB.BC, and the square of AC are equal to the gnomon AOH and the square XH, that is, the square of AD; and AD is equal to AB and BC added together in one straight line. Therefore, if a straight line, &c.

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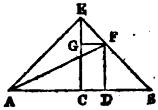
* Otherwise:—Because (II. 4) $AB^2 = AC^2 + BC^2 + 2\,AC.BC$, add BC^2 to both: then $AB^2 + BC^2 = AC^2 + 2\,BC^2 + 2\,AC.BC$. But (II. 3) $BC^2 + AC.BC = AB.BC$, and therefore $2\,BC^2 + 2\,AC.BC = 2\,AB.BC$; wherefore $AB^2 + BC^2 = AC^2 + 2\,AB.BC$.

† Though what is here asserted by Euclid is sufficiently obvious, yet, in reality, in making this assertion, the truth of the first proposition of the fifth book is

PROP. IX. THEOR.—If a straight line be divided into two equal, and also into two unequal parts; the squares of the unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the straight line AB be divided equally in C, and unequally in D: the squares of AD, DB are together double of the squares of AC, CD.



From C draw (I. 11) CE at right angles to AB, and make it equal to AC or CB, and join EA, EB: through D draw (I. 31) DF parallel to CE, and through F draw FG parallel to AB; and join AF. Then because (const.) the triangles ACE, BCE are right angled and isosceles, the angles CAE, AEC, CEB, EBC are (I. 32, cor. 4) each half a right angle. So also (I. 29, part 2) are EFG, BFD, because FG is parallel to AB, and FD to EC: and for the same reason, EGF, ADF are right angles. The angle AEB is also a right angle, its parts being each half a right angle. Hence (I. 6) EG is equal to GF or CD, and FD to DB. Again (I. 47) the square of AE is equal to the squares of AC, CE, or to twice the square of AC, since AC and CE are equal. In like manner, the square of EF is equal to twice the square of GF or CD. Now, (I. 47) the squares of AD and DF, or of AD and DB, are equal to the square of AF; and the squares of AE, EF, that is, twice the square of AC and twice the square of CD, are also equal to the square of AF: therefore (I. ax. 1) the squares of AD, DB are equal to twice the square

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assumed, without proof. In strictness, therefore, should any one follow out the tedious demonstration, here given, of a proposition which is in itself of scarcely any value, he ought first to demonstrate the proposition referred to. See also the note to the twentieth proposition of the third book. It may be farther remarked, that this proposition is virtually the same as the second corollary to the seventh proposition, AD being the sum, and AC the difference of AB and BC. Should any person wish to demonstrate it independently of that corollary, the following proof may probably be preferred to the tedious one given in the text.

$AD^2 = AC^2 + CD^2 + 2\,AC.CD$ (II. 4). But (II. 4, cor. 2) $CD^2 = 4\,CB^2$: also $2\,AC.CD = 4\,AC.CB$: wherefore $AD^2 = AC^2 + 4\,CB^2 + 4\,AC.CB$. But (II. 3) $AB.BC = CB + AC.CB$, and therefore $4\,AB.BC = 4\,CB^2 + 4\,AC.CB$. Hence $AD^2 = AC^2 + 4\,AB.BC$.

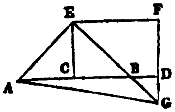
of AC and twice the square of CD. If, therefore, a straight line, &c.

OTHERWISE:—$DB^2+2BC.CD=BC^2+CD^2$ (II. 7), or $DB^2+2AC.CD=AC^2+CD^2$: also (II. 4) $AD^2=AC^2+CD^2+2AC.CD$. Add these equals together, and from the sums take 2 AC.CD; then $AD^2+DB^2=2AC^2+2CD^2$.

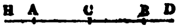
PROP. X. THEOR.—If a straight line be bisected, and produced to any point, the squares of the whole line thus produced, and of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D; the squares of AD, DB are double of the squares of AC, CD.

From C draw (I. 11) CE perpendicular to AB; and make it equal to AC or CB; join AE, EB, and through E and D draw (I. 31) EF parallel to AB, and DF parallel to CE. Then because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal (I. 29) to two right angles; and therefore the angles BEF, EFD are less than two right angles: therefore (I. ax. 12) EB, FD will meet, if produced towards B, D: let them meet in G, and join AG. Then it would be proved, as in the last proposition, that the angles CAE, AEC, CEB, EBC are each half a right angle, and AEB a right angle. BDG is also a right angle, being equal (I. 29) to ECB, since (const.) EC, FG are parallel; DGB, DBG are each half a right angle, being equal (I. 29) and (I. 15) to CEB, CBE, each to each; and FEG is half a right angle, being (I. 29) equal to CBE. It would also be proved, as in the last proposition, that the square of AE is twice the square of AC, and the square of EG twice the square of EF or CD. Now, (I. 47) the squares of AD, DG, or of AD, DB, are equal to the square of AG; and the squares of AE, EG, or twice the square of AC and twice the square of CD, are also equal to the square of AG. Therefore (I. ax. 1) the squares of AD, DB are equal to twice the square of AC and twice the square of CD. If, therefore, a straight line, &c.



OTHERWISE:—Produce CA, making CH equal to CD. To these add CB, CA; therefore HB, AD are equal. Then (II. 9) HB^2+BD^2, or $AD^2+BD^2=2CD^2+2AC^2$.*



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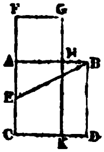
* The ten foregoing propositions may all be proved very easily by means of algebra, in connexion with the principles of mensuration, already established in the corollaries to the 46th proposition of the first book. Thus, to prove the fourth proposition, let $AC=a$, $CB=b$, and, consequently, $AB=a+b$. Now, the area of the square described on AB will be found (I. 46, cor. 4) by multiplying $a+b$ by itself. This product is found, by performing the actual operation, to be $a^2+2ab+b^2$; an expression, the first and third parts of which are, by the same corollary, the

Prop. XI. Prob.—To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, may be equal to the square of the other part.

Let AB be the given straight line; it is required to divide it into two parts, so that the rectangle under the whole and one of the parts, may be equal to the square of the other.

Upon AB describe (I. 46) the square AD; bisect (I. 10) AC in E, and join E with B, the remote extremity of AB; produce CA to F, making (I. 3) EF equal to EB, and cut off AH equal to AF: AB is divided in H, so that the rectangle AB.BH is equal to the square of AH.

Complete the parallelogram AG, and produce GH to K. Then, since BAC is a right angle, FAH is also (I. 13) a right angle; and (I. def. 27) AG is a square, AF, AH being equal by construction. Because the straight line AC is bisected in E, and produced to F, the rectangle CF.FA and the square of AE are together equal (II. 6) to the square of EF or of EB, since (const.) EB, EF are equal. But the squares of BA, AE are equal (I. 47) to the square of EB, because the angle EAB is a right angle: therefore the rectangle CF.FA and the square of AE are equal (I. ax. 1) to the squares of BA, AE. Take away the square of AE, which is common to both; therefore the remaining rectangle CF.FA is equal (I. ax. 3) to the square of AB. But the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the square of AB: therefore FK is equal to AD. Take away the common part AK, and (I. ax. 3) the remainders FH and HD are equal. But HD is the rectangle AB.BH, for AB is equal to BD; and FH is the square of AH. Therefore the rectangle AB.BH is equal to the square of AH: wherefore the straight line AB is divided in H, so that the rectangle AB.BH is equal to the square of AH:* which was to be done.



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areas of the squares of AC and CB, and the second is twice the rectangle of those lines.

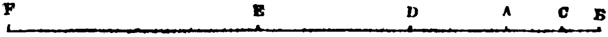
In like manner, to prove the eighth, adopting the same notation, we have the line which is made up of the whole and CB $= a+2b$; and, multiplying this by itself we get for the area of the square of that line, $a^2+4ab+4b^2$, or $a^2+4(a+b)b$, the first part of which is the area of the square of AC, and the second four times the area of the rectangle under AB and CB.

It will be a useful exercise for the student to prove the other propositions in a similar manner. He will also find it easy to investigate various other relations of lines and their parts by means of algebra.

All the properties delivered in these propositions hold also respecting numbers, if products be substituted for rectangles. Thus, 7 being equal to the sum of 5 and 2, the square, or second power, of 7, is equal to the squares of 5 and 2 and twice their product; that is, $49=25+4+20$.

* In the practical construction in this proposition, and in the 30th of the sixth book, which is virtually the same, it is sufficient to draw AE perpendicular to AB, making it equal to the half of AB, and producing it through A; then, to make EF equal to the distance from E to B, and AH equal to AF. It is plain that BD

Schol. The line CF is equal to BA and AH; and since it has been shown that the rectangle CF.FA is equal to the square of BA or CA, it follows, that if any straight line AB (see the next diagram) be divided according to this proposition in C, AC being the greater part, and if AD be made equal to AB, DC is similarly divided in A. So also if DE be made equal to DC, and EF to EA, EA is divided similarly in D, and FD in E: and the like additions may be continued as far as we please.

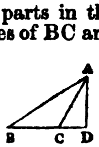


Conversely, if any straight line FD be divided according to this proposition in E, and if EA be made equal to EF, DC to DE, &c. EA is similarly divided in D, DC in A, &c. It follows also, that the greater segment of a line so divided, will be itself similarly divided, if a part be cut off from it equal to the less; and that by adding to the whole line its greater segment, another line will be obtained, which is similarly divided.

PROP. XII. THEOR.—In an obtuse-angled triangle, the square of the greatest side exceeds the squares of the other two, by twice the rectangle contained by either of the last-mentioned sides, and its continuation to meet a perpendicular drawn to it from the opposite angle.

Let ABC be a triangle, having the angle ACB obtuse; and let AD be perpendicular to BC produced; the square of AB is equal to the squares of AC and CB, and twice the rectangle BC.CD.

Because the straight line BD is divided into two parts in the point C, the square of BD is equal (II. 4) to the squares of BC and CD, and twice the rectangle BC.CD. To each of these equals add the square of DA; and the squares of DB, DA are equal to the squares of BC, CD, DA, and twice the rectangle BC.CD. But, because the angle D is a right angle, the square of BA is equal (I. 47) to the squares of BD, DA, and the



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might be bisected instead of AC, and that in this way another point of section would be obtained.

While the enunciation in the text serves for ordinary purposes, it is too limited in a geometrical sense, as it comprehends only one case, excluding another. The following includes both:

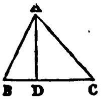
In a given straight line, or its continuation, to find a point, such that the rectangle contained by the given line, and the segment between one of its extremities and the required point, may be equal to the square of the segment between its other extremity and the same point.

The point in the continuation of BA, will be found by cutting off a line on EC and its continuation, equal to EB, and describing on the line composed of that line and AE, a square lying on the opposite side of AC from AD; as the angular point of that square in the continuation of BA is the point required. The proof is the same as that given above, except that a rectangle corresponding to AK is to be added instead of being subtracted.

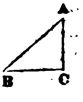
square of CA is equal to the squares of CD, DA; therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC.CD. Therefore, in an obtuse-angled triangle, &c.

Prop. XIII. Theor.—In any triangle, the square of a side subtending an acute angle, is less than the squares of the other sides, by twice the rectangle contained by either of those sides, and the straight line intercepted between the acute angle and the perpendicular drawn to that side from the opposite angle.

Let ABC (see this figure and that of the foregoing proposition) be any triangle, having the angle B acute; and let AD be perpendicular to BC, one of the sides containing that angle: the square of AC is less than the squares of AB, BC, by twice the rectangle CB.BD.



The squares of CB, BD are equal (II. 7) to twice the rectangle contained by CB, BD, and the square of DC. To each of these equals add the square of AD: therefore the squares of CB, BD, DA are equal to twice the rectangle CB.BD, and the squares of AD, DC. But, because AD is perpendicular to BC, the square of AB is equal (I. 47) to the squares of BD, DA, and the square of AC to the squares of AD, DC: therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB.BD; that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB.BD.



If the side AC be perpendicular to BC, then BC is the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares of AB, BC are equal (I. 47) to the square of AC and twice the square of BC. Therefore, in any triangle, &c.[*]

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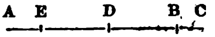
[*] By means of this or the foregoing proposition, the area of a triangle may be computed, if the sides be given in numbers. Thus, let $AB=17$, $BC=28$, and $AC=25$. From AB^2+BC^2 take AC^2; that is, from 17^2+28^2 take 25^2; the remainder 448 is twice CB.BD. Dividing this by 56, twice BC, the quotient 8 is BD. Hence, from either of the triangles ABD, ACD, we find the perpendicular AD to be 15; and thence the area is found, by taking half the product of BC and AD, to be 210.

The segments of the base are more easily found by means of the 4th corollary to the 5th proposition of this book, in connexion with the principle, that *if half the difference of two magnitudes be added to half their sum, the result is the greater; and if half the difference be taken from half the sum, the remainder is the less.* Thus, if 42, the sum of AB and AC, be multiplied by 8, their difference, and if the product 336 be divided by 28, the sum of the segments of the base, the quotient 12 is their difference. The half of this being added to the half of 28, the sum 20 is the greater segment CD; and being subtracted from it, the remainder 8 is BD.

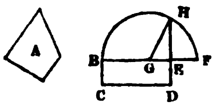


To prove the principle mentioned above, let AB be the greater, and BC the less of two magnitudes. Bisect AC in D, and make AE equal to BC. Then AD or DC is half the sum, and ED or DB half the difference of AB and BC; and AB the greater is equal to the sum of AD and DB, and BC the less is equal to the difference of DC and DB.

Prop. XIV. Prob.*—To describe a square that shall be equal to a given rectilineal figure.

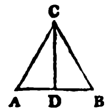
Let A be the given rectilineal figure; it is required to describe a square that shall be equal to it.

Describe (I. 45) the rectangle BD equal to A. Then, if the sides of it, BE, ED, be equal to one another, it is a square, and what was required is done. But if they be not equal, produce one of them BE to F, and make EF equal to ED: bisect BF in G, and from the centre G, at the distance GB, or GF, describe (I. post. 3) the semicircle BHF: produce DE to H, and join GH. Therefore, because the straight line BF is divided equally in G, and unequally in E, the rectangle BE.EF, and the square of EG, are equal (II. 5) to the square of GF, or of GH, because GH is equal to GF. But the squares of HE, EG are equal (I. 47) to the square of GH: therefore the rectangle BE.EF and the square of EG are equal to the squares of HE, EG. Take away the square of EG, which is common, and the remaining rectangle BE.EF is equal to the square of EH. But the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED: therefore BD is equal to the square of EH: but BD is equal to the figure A: therefore the square of EH is equal to A. The square described on EH, therefore, is the required square.



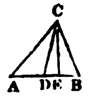
Prop. A. Theor.—The squares of two sides of a triangle are together equal to twice the square of half the remaining side, and twice the square of the straight line drawn from its point of bisection to the opposite angle.

Let ABC be a triangle, of which the side AB is bisected in D, and let DC be drawn to the opposite angle; the squares of AC and CB are together double of the squares of AD and DC.



If DC be perpendicular to AB, the squares of AC and CB are equal (I. 47) to the squares of AD and DB and twice the square of DC, or to twice the square of AD and twice the square of DC, because AD, DB are (I. 47, cor. 5) equal.

But if CD be not perpendicular to AB, draw CE perpendicular to it. Then (II. 12) the square of AC is equal to the squares of AD and DC, together with twice the rectangle AD.DE; and (II. 13) the square of CB and twice the rectangle BD.DE, or AD.DE, are together equal to the squares of BD and CD, or of AD and DC. Add these equals together, and the squares of AC and CB, together with twice the rectangle AD.DE, are equal to twice the squares of AD and DC, together



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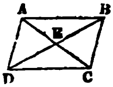
* This is a particular case of the 25th proposition of the sixth book.

with twice the rectangle AD.DE. Take away twice the rectangle AD.DE, and the squares of AC and CB are equal to twice the squares of AD and DC. Therefore the squares, &c.*

Prop. B. Theor.—The squares of the diagonals of a parallelogram are together equal to the squares of its sides.

Let ABCD be a parallelogram, of which the diagonals are AC and BD: the sum of the squares of AC and BD is equal to the sum of the squares of the sides AB, BC, CD, DA.

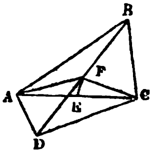


Let AC and BD intersect each other in E. Then, in the triangles AED, CEB, the angles AED, ADE are respectively equal (I. 15, and 29) to CEB, CBE, and (I. 34) the side AD is equal to CB: therefore the sides AE, ED are respectively equal (I. 26) to CE, EB. Since, therefore, BD is bisected in E, the squares of AB, AD are together equal (II. A) to twice the squares of AE, EB. But BC and CD are respectively equal (I. 34) to AD and AB; and therefore the squares of AB, BC, CD, and DA are together equal to four times the squares of AE and EB, that is (II. 4, cor. 2), to the squares of AC and BD, since AC is double of AE, and BD of BE. Therefore, the squares, &c.†

Cor. The diagonals of a parallelogram bisect one another.

Prop. C. Theor.—The sum of the squares of the sides of a trapezium is equal to the sum of the squares of the diagonals, together with four times the square of the straight line joining the points of bisection of the diagonals.

Let ABCD be a trapezium, having its diagonals AC, BD bisected in E and F, and let EF be joined: the squares of AB, BC, CD, DA are together equal to the squares of AC, BD, together with four times the square of EF.



Join AF, FC. The squares of AB, AD are together equal (II. A) to twice the sum of the squares of DF and AF; and the squares of BC, CD are equal to twice the sum of the squares of DF and CF. Add these equals together, and the sum of the squares of AB, BC, CD, DA is equal to four times the square of DF, together with twice the sum of the squares of AF and CF. But twice the squares of AF and CF are equal

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* Hence, if the sides of a triangle be given in numbers, the line AD can be computed. Thus, if AC=11, AB=14, and BC=7, we have $AC^2+BC^2=121+49=170$, and $2AD^2=98$. Then 170—98=72, the half of which is 36; and 6, the square root of this, is CD.

† Hence, if we have the sides and one of the diagonals of a parallelogram in numbers, we can compute the remaining diagonal. Thus, if AB, DC be each =9, AD, BC each=7, and AC=8, we have $AB^2+BC^2+CD^2+DA^2=81+49+81+49=260$, and $AC^2=64$. Taking the latter from the former, and extracting the square root, we find BD=14.

(II. A) to four times the squares of AE and EF; and (II. 4, cor. 2) four times the square of DF is equal to the square of BD, and four times the square of AE to the square of AC. Hence the squares of AB, BC, CD, DA are equal to the squares of AC and BD, together with four times the square of EF. Therefore the sum, &c.*

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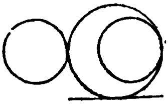
[*] Proposition B may be regarded as a particular case of this proposition, the line EF vanishing in the parallelogram, since the diagonals bisect each other; and thus the proof of the former may be regarded as being contained in this. In all such cases, however, Euclid gives separate proofs; and it seems better to do so in an elementary work.

BOOK III.[*]

DEFINITIONS.[†]

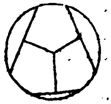
1. A STRAIGHT line is said to *touch* a circle, or to be a *tangent* to it, when it meets the circle, and being produced does not cut it.

2. Circles are said to *touch* one another, which meet, but do not cut one another.



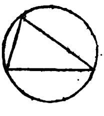
3. In a circle, chords are said to be *equally distant from the centre*, when the perpendiculars drawn to them from the centre are equal:

4. And the chord on which the greater perpendicular falls, is said to be *farther from the centre.*

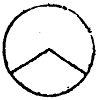


5. An *angle in a segment of a circle* is an angle contained by two straight lines drawn from any point in the arc of the segment to the extremities of its chord:

6. And an angle is said to *stand upon* the arc intercepted between the straight lines that contain the angle.



7. A *sector* of a circle is a figure contained by any arc of the circle, and two radii drawn through its extremities.[‡]



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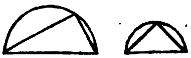
[*] The third book of the Elements treats of the properties of the circle, and of lines and angles dependent on it.

[†] The first definition in Simson's edition is as follows:—"Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal." This is here omitted, because, as Simson justly remarks, it "is not a definition but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal." The definition of a segment is also omitted, as it is given in the first book.

[‡] As, by the second proposition of this book, any chord divides a circle into two segments, so any two radii divide it into two sectors.

A *quadrant* is a sector whose radii are perpendicular to each other. It is easy to show by superposition, that a quadrant is half of a semicircle, and therefore a fourth part of the entire circle.

8. *Similar sequents* of circles are those which contain equal angles.



9. *Concentric* circles are those which have the same centre.

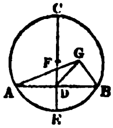
PROPOSITIONS.

PROP. I. PROB.—To find the centre of a given circle.

Let ABC be a given circle; it is required to find its centre.

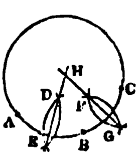
Draw any chord AB, and (I. 10) bisect it in D; through D draw (I. 11) the chord EDC at right angles to AB, and bisect it in F: F is the centre.*

For, if it be not, let, if possible, G, a point not in CE, be the centre, and join GA, GD, GB. Then, in the triangles ADG, BDG, because DA is equal (const.) to DB, DG common, and GA equal (I. def. 30) to GB, because they are drawn from the centre G to the circumference: therefore (I. 8) the angle ADG is equal to GDB. But (I. def. 8) when one straight line standing on another, makes the adjacent angles equal, each of the angles is a right angle: therefore GDB is a right angle; but (const.) FDB is likewise a right angle: wherefore FDB is equal to GDB, the greater to the less, which is impossible: therefore G is not the centre of the circle. In the same manner it can be shown, that no other point, which is not in CE, is the centre: the centre is therefore in CE: and since FC is equal to FE, F is the centre: which was to be found.†



Cor. From this it is manifest, that a straight line which bisects a chord of a circle at right angles, passes through the centre.

Schol. The practical construction is very simply effected in the following manner: Take any three points A, B, C, in the circumference, and from A and B as centres, with any radius greater than half their distance, describe arcs intersecting each other in D and E; also from B and C as centres, with any radius greater than half their distance, describe arcs intersecting in F and G: through E and D, and through G and F, draw straight lines intersecting each other in H: H is the centre. The proof is evident from the preceding corollary; for EH and GH are evidently perpendicular to chords joining AB and



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* In Simson's edition, the construction commences thus, "Draw *within* the circle any straight line AB." This mode of expression is improper, as it is not to be assumed till the second proposition is proved, that AB falls *within* the circle. In like manner, we are directed to *produce* DC to E, which also implies what is proved in the second proposition.

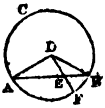
† The demonstration of this proposition which is given in the editions of Simson, Playfair, and others is imperfect; as, according to it, any point whatever in CE, or its continuations, would appear to be the centre.

BC, and bisect them, and therefore the centre must be in both EH and GH; it must therefore be their point of intersection, which is the only point common to both.

PROP. II. THEOR.—The straight line which joins any two points in the circumference of a circle, falls within the circle.

Let ABC be a circle, and A, B any two points in the circumference; the chord drawn from A to B falls within the circle.

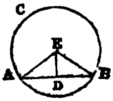
Take any point in AB as E: find (III. 1) D the centre of the circle ABC; join AD, DB, DE, and let DE meet the circumference in F. Then, because DA is equal to DB, the angles DAB, DBA are (I. 5) equal; and because AE, a side of the triangle DAE, is produced to B, the angle DEB is greater (I. 16) than DAE; but DAE is equal to DBE: therefore the angle DEB is greater than DBE. Now (I. 19) to the greater angle the greater side is opposite; DB is therefore greater than DE: but DB is equal to DF; wherefore DF is greater then DE, and the point E is therefore within the circle. The same may be demonstrated of any other point between A and B; therefore AB is within the circle. Wherefore the straight line, &c.



PROP. III. THEOR.—If a straight line drawn from the centre of a circle bisect a chord, which does not pass through the centre, it cuts it at right angles; and (2) if it cut it at right angles it bisects it.

Let ABC be a circle; and let ED, a straight line drawn from the centre E, bisect any chord AB, which does not pass through the centre, in the point D; ED cuts AB at right angles.

Join EA, EB. Then in the triangles ADE, BDE, AD is equal to DB, DE common, and (I. def. 30) the base EA is equal to the base EB; therefore (I. 8) the angles ADE, BDE are equal; and consequently (I. def. 8) each of them is a right angle; wherefore ED cuts AB at right angles.

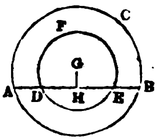


Next, let ED cut AB at right angles; ED also bisects it.

The same construction being made, because the radii EA, EB are (I. def. 30) equal, the angle EAD is equal (I. 5) to EBD; and the right angles ADE, BDE are equal; therefore, in the two triangles EAD, EBD, there are two angles in one equal to two angles in the other, each to each, and the side ED, which is opposite to one of the equal angles in each, is common; therefore (I. 26) AD is equal to DB. If a straight line, therefore, &c.

Cor. 1. Hence, in an isosceles triangle, a straight line drawn from the vertex bisecting the base, is perpendicular to it; and a straight line drawn from the vertex perpendicular to the base, bisects it.

Cor. 2. Let the straight line AB cut the concentric circles ABC, DEF in the points A, D, E, B; AD is equal to EB, and AE to DB. From the common centre G, draw GH perpendicular to AB. Then (III. 3) AH is equal to HB, and DH to HE. From AH take DH, and from HB take HE, and the remainders AD, EB are equal. To these equals add DE, and the sums AE, DB are equal.

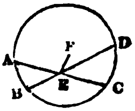


Cor. 3. Any number of parallel chords in a circle are all bisected by a diameter perpendicular to them.

Prop. IV. Theor.—Two chords of a circle, which are not both diameters, cannot bisect each other.

Let ABCD be a circle, and let AC, BD, two chords in it, which are not both diameters, cut one another in the point E: they do not bisect one another.

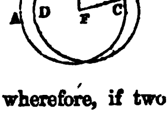
For, if it be possible, let AE be equal to EC, and BE to ED. If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre. But if neither of them pass through the centre, take (III. 1) F the centre, and join EF; and because FE, a straight line drawn through the centre, bisects the chord AC, which does not pass through the centre, it cuts it (III. 3) at right angles; wherefore FEA is a right angle. Again, because FE bisects BD, which does not pass through the centre, FEB is also (III. 3) a right angle: and FEA was shown to be a right angle: therefore (I. ax. 11) FEA is equal to FEB, which (I. ax. 9) is impossible. Therefore AC, BD do not bisect one another. Wherefore two chords, &c.



Prop. V. Theor.—If two circles cut one another, they have not the same centre.

Let the two circles ABC, DBE cut one another in the point B; they have not the same centre.

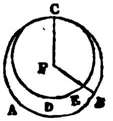
For, if it be possible, let F be their centre: join FB, and draw any straight line FCE meeting the circumferences in C and E: then (I. def. 30) because F is the centre of the circle ABC, FB is equal to FC; and, because F is the centre of the circle DBE, FB is equal to FE. But FB was shown to be equal to FC: therefore FE is equal (I. ax. 1) to FC, which (I. ax. 9) is impossible. Therefore F is not the centre of the circles ABC, DBE: wherefore, if two circles, &c.



Prop. VI. Theor.—If one circle touch another internally, they have not the same centre.

Let the circle ABC be touched internally by the circle DEC in the point C: they have not the same centre.

For, if they can, let it be F; join FC, and draw any straight line FEB meeting the circumferences in E and B. Then (I. def. 30) because F is the centre of the circle ABC, FC is equal to FB; and, because F is the centre of the circle CDE, FC is equal to FE. But FC has been shown to be equal to FB: therefore FE is equal to FB, which (I. ax. 9) is impossible; wherefore F is not the centre of the circles ABC, CDE; and in the same manner it can be shown, that no other point can be the centre of both the circles. Therefore, if one circle, &c.



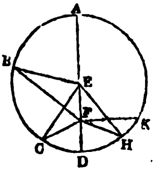
PROP. VII. THEOR.—If from any point within a circle, which is not the centre, straight lines be drawn to the circumference: (1) the greatest is that which passes through the centre, and (2) the continuation of that line to the circumference, in the opposite direction, is the least: (3) of others, one nearer to the line passing through the centre is greater than one more remote: and (4) from the same point there can be drawn only two equal straight lines, one upon each side of either the longest or shortest line, and making equal angles with that line.

Let ABCD be a circle. E its centre, and AD a diameter, in which let any point F be taken, which is not the centre: of all the straight lines FA, FB, FC, &c., that can be drawn from F to the circumference, FA is the greatest, and FD the least: and of the others, FB is greater than FC.

1. Join BE, CE. Then (I. 20) BE, EF are greater than BF; but AE is equal to EB; therefore AF, that is, AE, EF, is greater than BF.

2. Because CF, FE are greater (I. 20) than EC, and EC is equal to ED; CF, FE are greater than ED. Take away the common part FE, and (I. ax. 5) the remainder CF is greater than the remainder FD.

3. Again, because BE is equal to CE, and FE common to the triangles BEF, CEF; but the angle BEF is greater than CEF: therefore (I. 24) the base BF is greater than the base CF.



4. Make (I. 23) the angle FEH equal to FEC, and join FH. Then, because CE is equal to HE, EF common to the two triangles CEF, HEF, and the angle CEF equal to the angle HEF; therefore (I. 4) the base FC is equal to the base FH, and the angle EFC to the angle EFH. But, besides FH, no other straight line can be drawn from F to the circumference equal to FC. For, if there can, let it be FK; and because FK is equal to FC, and FC to FH, FK is equal (I. ax. 1) to FH; that is, a line nearer to

that which passes through the centre, is equal to one which is more remote; which is impossible. Therefore, if from any point, &c.

PROP. VIII. THEOR.—If from any point without a circle straight lines be drawn to the circumference; (1) of those which fall upon the concave part of the circumference, the greatest is that which passes through the centre; and (2) of the rest, one nearer to the greatest is greater than one more remote. (3) But of those which fall upon the convex part, the least is that which when produced, passes through the centre; and (4) of the rest, one nearer to the least is less than one more remote. And (5) only two equal straight lines can be drawn from the point to either part of the circumference, one upon each side of the line passing through the centre, and making equal angles with it.

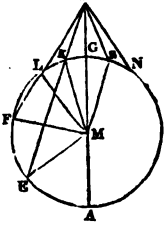
Let ABF be a circle, M its centre, and D any point without it, from which let the straight lines DA, DE, DF be drawn to the circumference. Of those which fall upon the concave part of the circumference AEF, the greatest is DMA, which passes through the centre; and a line DE nearer to it is greater than DF, one more remote. But of those which fall upon the convex circumference LKG, the least is DG, the external part of DMA; and a line DK nearer to it is less than DL, one more remote.

1. Join ME, MF, ML, MK; and because MA is equal to ME, add MD to each; therefore AD is equal to EM, MD: but (I. 20) EM, MD are together greater than ED; therefore also AD is greater than ED.

2. Because ME is equal to MF, and MD common to the triangles EMD, FMD, but the angle EMD is greater than FMD; therefore (I. 24) the base ED is greater than the base FD.

3. Because (I. 20) MK, KD are greater than MD, and MK is equal to MG, the remainder KD is greater (I. ax. 5) than the remainder GD; that is, GD is less than KD.

4. Because MK is equal to ML, and MD common to the triangles KMD, LMD, but the angle DMK less than DML; therefore the base DK is less (I. 24) than the base DL.



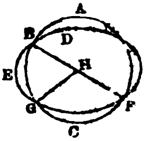
5. Make (I. 23) the angle DMB equal to DMK, and join DB. Then, because MK is equal to MB, MD common to the triangles KMD, BMD, and the angle KMD equal to BMD; therefore (I. 4) the base DK is equal to the base DB, and the angle MDK to the angle MDB. But, besides DB, there can be no straight line drawn from D to the circumference equal to DK: for, if there can, let it be DN; and because DK is equal to DN, and also to DB;

therefore DB is equal to DN; that is, a line nearer to the least equal to one more remote, which is impossible. If from any point, therefore, &c.

Prop. IX. Theor.—A point in a circle from which more than two equal straight lines fall to the circumference, is the centre.

For, from any point which is not the centre, only two equal straight lines (III. 7) can be drawn to the circumference, and therefore a point from which more than two equal straight lines can be drawn to the circumference, cannot be any other than the centre. Therefore a point, &c.*

Prop. X. Theor.—One circle cannot cut another in more than two points.



If it be possible, let the circle FAB cut the circle DEF in more than two points, viz. in B, G, F: take (III. 1) the centre H, of the circle ABC, and join HB, HG, HF. Then, because within the circle DEF there is taken the point H, from which there fall to the circumference DEF more than two equal straight lines HB, HG, HF, H is (III. 9) the centre of the circle DEF; but H is also the centre of the circle ABC: therefore the same point is the centre of two circles that cut one another, which (III. 5) is impossible. Therefore one circle, &c.

Prop. XI. Theor.—If one circle touch another internally in any point, the straight line which joins their centres, being produced, passes through that point.†

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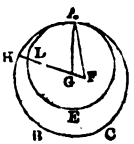
* The following is an outline of a direct demonstration of this proposition: Let M in the diagram for the last proposition be a point from which the equal straight lines, MA, ME, MF, are drawn to the circumference. Then, if a straight line be drawn from M to the point of bisection of the chord joining AE, it will be perpendicular (hyp. and I. 8) to the chord; and, therefore (III. 1, cor.) the centre will be in that perpendicular. In a similar manner, it would be shown, that the centre would be in the straight line drawn from M to the point of bisection of the chord joining EF. Hence, therefore, M, the point common to the two perpendiculars, must be the centre.

† Dr. Simson's enunciation is as follows:—"If two circles touch each other internally, the straight line which joins their centres, being produced, shall pass through the point of contact." This is liable to two objections. The circles do not "touch *each other* internally," the interior one being touched *externally*. The expression, "*the* point of contact," seems to assume that there is but one point of contact, which is proved in a following proposition, the 13th. Mr. Walker proposes to say, *a* point of contact; but this would seem to imply that there may be more points of contact than one. In the enunciation here given, these objections are obviated. A similar change is made in one part of the enunciation of the 12th proposition, and also of the 6th.

With regard to this proposition, since, till the 13th proposition is proved, we are not to assume that the circles touch one another in only one point, we ought to consider the case, in which, if possible, the continuation of FG would pass through

Let the circle ADE touch the circle ABC internally in the point A, and let F be the centre of ABC, and G the centre of ADE: the straight line which joins F and G, being produced, passes through A.

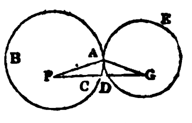
For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG. Then, because (I. 20) AG, GF are greater than FA, that is, than FH, for FA is equal to FH, each being a radius of ABC; take away the common part FG: therefore (I. ax. 5) the remainder AG is greater than the remainder GH; but AG is equal (I. def. 30) to GD: therefore GD is greater than GH, which (I. ax. 9) is impossible. The straight line, therefore, which joins the points F, G cannot fall otherwise than upon A, that is, it must pass through it. Therefore, if one circle, &c.



Prop. XII. Theor.—If two circles touch each other externally in any point, the straight line which joins their centres passes through that point.

Let the two circles ABC, ADE touch each other externally in a point A; and let F and G be their centres: the straight line which joins F and G passes through A.

For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG. Then (I. def. 30) because F and G are the centres of the circles, AF is equal to FC, and AG to GD: therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG, which (I. 20) is impossible: therefore the straight line which joins the points F, G cannot pass otherwise than through the point A: it therefore passes through that point: wherefore, if two circles, &c.



Prop. XIII. Theor.—One circle can touch another in only one point, whether it touch it internally or externally.

For, by the two preceding propositions, if two circles touch one another in any point, the straight line joining their centres passes through that point. If therefore the circles could touch one another in more points than one, their centres would lie in the straight line joining two such points. Now, that straight line would be a chord of each of the circles; and, therefore (III. 1, cor.) the centres would also lie in a straight line bisecting it, and perpendicular to

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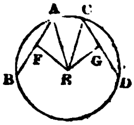
another point of contact, so that D and H would coincide. In this case, we should still have, as in the first part of the demonstration, AG greater than GH or GD; while, from the coincidence of D and H, GA and GD or GH must be equal. Similar remarks are applicable with regard to the next proposition.

it. Hence, through the centres there would pass two straight lines not coinciding, which (I. def. 3) is impossible. One circle, therefore, &c.

Prop. XIV. Theor.—Equal chords in a circle are equally distant from the centre; and (2) chords which are equally distant from the centre are equal to one another.

Let the chords AB, CD, in the circle ABDC, be equal to one another; they are equally distant from the centre.

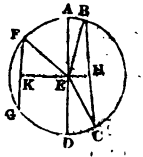
Take (III. 1) E the centre of the circle, and draw (I. 12) EF, EG perpendiculars to AB, CD: join also EA, EC. Then, because the straight line EF, passing through the centre, cuts the chord AB, which does not pass through the centre, at right angles, it also (III. 3) bisects it: wherefore AF is equal to FB, and AB is double of AF. For the same reason, CD is double of CG; but AB is equal to CD; therefore AF is equal (I. ax. 7) to CG. Then, in the right-angled triangles EFA, EGC, the sides EA, AF are equal to the sides EC, CG, each to each; therefore (I. 47, cor. 5) the sides EF, EG are equal. But chords in a circle are said (III. def. 3) to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal: therefore AB, CD are equally distant from the centre.



Next, if the chords AB, CD be equally distant from the centre, that is, if FE be equal to EG, AB is equal to CD. For, the same construction being made, it may, as before, be demonstrated, that AB is double of AF, and CD of CG; and because the right-angled triangles EFA, EGC have the sides AE, EF equal to CE, EG, each to each, the sides AF, CG are also (I. 47, cor. 5) equal to one another. But AB is double of AF, and CD of CG; wherefore AB is equal (I. ax. 6) to CD. Therefore equal chords, &c.

Prop. XV. Theor.—The diameter of a circle is the greatest chord: (2) of others, one nearer to the centre is greater than one more remote; and (3) the greater is nearer to the centre than the less.

Let ABCD be a circle, of which the diameter is AD, and the centre E: and let BC be nearer to the centre than FG; AD is greater than BC, and BC than FG.



1. From the centre draw (I. 12) EH, EK perpendiculars to BC, FG, and join EB, EC, EF. Then, because AE is equal to EB, and ED to EC, AD is equal to EB, EC: but EB, EC are greater (I. 20) than BC; wherefore also AD is greater than BC.

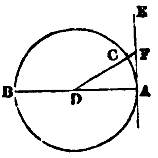
2. Again, because BC is nearer to the centre than FG, EH (III. def. 4) is less than EK; and the hypotenuses EB, EF are equal. Therefore, (I. 47, cor. 6)

BH is greater than FK. But, as was demonstrated in the preceding proposition, BC is double of BH, and FG of FK: therefore BC is greater than FG.

3. Next, let BC be greater than FG; BC is nearer to the centre than FG; that is, the same construction being made, EH is less than EK. Because BC is greater than FG, BH likewise is greater than FK: and therefore (I. 47, cor. 6) EH is less than EK. Therefore the diameter, &c.

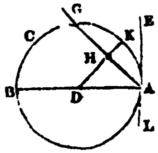
PROP. XVI. THEOR.—The straight line drawn at right angles to a diameter of a circle, through its extremity, falls without the circle; but any other straight line drawn through that point cuts the circle.

Let ABC be a circle, of which D is the centre, and AB a diameter: if AE be drawn through A perpendicular to AB, it falls without the circle.



In AE take any point F; and draw DF, meeting the circumference in C. Because DAF is a right angle, it is greater (I. 17) than DFA; and therefore (I. 19) DF is greater than DA. But (I. def. 30) DA is equal to DC; therefore DF is greater than DC, and the point F is therefore without the circle: and in the same manner it may be shown, that any other point in AE, except the point A, is without the circle.

Again, any other straight line drawn through A, cuts the circle. Let AG be drawn in the angle DAE, and draw (I. 12) DH perpendicular to AG, and meeting the circumference in K. Then, because DHA is a right angle, and DAH less than a right angle, being a part of DAE, the side DH is less (I. 19) than the side DA. But (I. def. 30) DK is equal to DA; therefore DH is less than DK; the point H is therefore within the circle; and AG cuts the circle, since its continuation through A, must fall on the opposite side of EAL, and must therefore be without the circle. Therefore the straight line, &c.

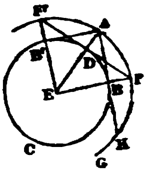


Cor. From this it is manifest, that the straight line which is drawn at right angles to a diameter of a circle from its extremity, touches (III. def. 1) the circle; and that it touches it only in one point, because at every point except A, it falls without the circle. It is also evident, that there can be but one tangent at the same point of a circle.

PROP. XVII. PROB.—From a given point, either without a given circle, or in its circumference, to draw a straight line touching the circle.

First, let A be a given point without the given circle BCD; it is required to draw from A a straight line touching the circle.

Find (III. 1) E the centre of the circle, and draw AE cutting the circumference in D; from the centre E, at the distance EA, describe (I. post. 3) the circle AFG; from D draw (I. 11) DF at right angles to EA; and draw EBF, and join AB. AB touches the circle BCD.



Because E is the centre of the circles, EA is equal to EF, and ED to EB; therefore the two sides AE, EB are equal to the two FE, ED, each to each, and they contain the angle AEF common to the two triangles AEB, FED: therefore the angle EBA is equal (I. 4, part. 3) to EDF, and is, therefore, a right angle, because (const.) EDF is a right angle. Now, since EB is drawn from the centre, it is part of a diameter of which B is one extremity: but a straight line drawn from the extremity of a diameter, at right angles to it touches (III. 16, cor.) the circle; therefore AB touches the circle; and it is drawn from the given point A: which was to be done.

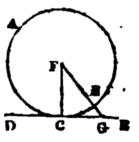
Secondly, if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; DF touches (III. 16, cor.) the circle.

Cor. If AB be produced to H, AH is bisected (III. 3) in B. Hence a chord in a circle touching a concentric one, is bisected at the point of contact.

Schol. It is evident that from any point A without the circle, two tangents, AB and AB′, may be drawn to the circle, and that these are equal to one another, being equal respectively to the equal lines DF and DF′.*

PROP. XVIII. THEOR.—If a straight line touch a circle, the straight line drawn from the centre to the point of contact, is perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in C; take the centre F, and join FC: FC is perpendicular to DE.



For, if it be not, from F draw FG perpendicular to DE, meeting the circle in B. Then (III. 16, cor.) FG is greater than FB: and (I. 17) because FGC is a right angle, GCF is an acute angle: therefore (I. 19) FC is greater than FG; but (I. def. 30) FC is equal to FB; therefore FB is greater than FG; which (I. ax. 9) is impossible:

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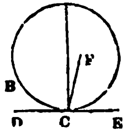
* The construction of the first case of this problem is as easily effected in practice, by describing a circle on AE as diameter, as its circumference will cut that of the given circle in the points B and B′. The reason of this will be evident from the 31st proposition of this book.

wherefore FG is not perpendicular to DE. In the same manner it may be shown that no other straight line drawn from F, except FC, is perpendicular to DE. Therefore, if a straight line, &c.

PROP. XIX. THEOR.—If a straight line touch a circle, a straight line drawn from the point of contact, at right angles to the tangent, passes through the centre.

Let the straight line DE touch the circle ABC in C, and let CA be perpendicular to DE; the centre of the circle is in CA.

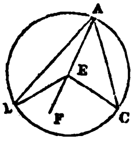
For, if not, let F be the centre, if possible, and join CF. Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular (III. 18) to DE: therefore FCE is a right angle; but ACE is also (hyp.) a right angle; therefore FCE is equal to ACE, which (I. ax. 9) is impossible: wherefore F is not the centre of the circle ABC. In the same manner it may be shown, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a straight line, &c.



PROP. XX. THEOR.—The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

In the circle ABC, let BEC be an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base; BEC is double of BAC.

Draw AE, and produce it to F; and first, let E, the centre of the circle, be within the angle BAC. Because EA is equal to EB, the angle EAB is equal (I. 5) to EBA: therefore the angles EAB, EBA are together double of EAB: but (I. 32) the angle BEF is equal to EAB, EBA; therefore also BEF is double of EAB. For the same reason, the angle FEC is double of the angle EAC; therefore the whole angle BEC is double of the whole BAC.*



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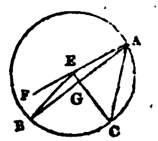
* *That, if two magnitudes be double of two others, each of each, the sum and difference of the first two are respectively double of the sum and difference of the other two*, follows from the first and fifth propositions of the fifth book. It is also thus proved by Playfair: "Let A and B, C and D be four magnitudes, such that $A=2C$, and $B=2D$; then $A+B=2(C+D)$. For since $A=C+C$, and $B=D+D$, adding equals to equals, $A+B=(C+D)+(C+D)=2(C+D)$. So also, if A be greater than B, and therefore C greater than D, since $A=C+C$, and $B=D+D$, taking equals from equals, $A-B=(C-D)+(C-D)$, that is $A-B=2(C-D)$."

The following is an outline of another proof of the second case: from the triangle BGE we have (I. 32) $BGC=BEC+B=BEC+EAG$ (I. 5). We have also, from the triangle ACG, in a similar manner, $BGC=BAC+C=BAC+EAC=2BAC+EAG$. Hence (I. ax. 1) $BEC+EAG=2BAC+EAG$. Take away EAG, &c.

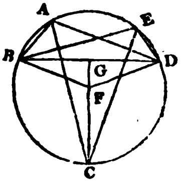
Again, let E the centre of the circle be without the angle BAC: it may be demonstrated, as in the first case, that the angle FEC is double of FAC, and that FEB, a part of the first, is double of FAB a part of the other; therefore the remaining angle BEC is double of the remaining angle BAC. The angle at the centre, therefore, &c.[*]



Prop. XXI. Theor.—Angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED: these angles are equal.

Take (III. 1) F the centre of the circle ABCD; and join BF, FD: draw also FG perpendicular to BD, and produce it to meet the circumference in C; and join AC, EC. Then, because the angle BFC is at the centre, and the angle BAC at the circumference, and that they have the same arc BC for their base: therefore (III. 20) the angle BFC is double of the angle BAC. For the same reason, the angle BFC is double of the angle BEC: therefore the angle BAC is equal (I. ax. 7) to the angle BEC. In the same manner, it might be shown that the angles CAD, CED are equal. Therefore (I. ax. 2) the whole angle BAD is equal to the whole angle BED. Wherefore angles, &c.[†]



Prop. XXII. Theor.—The opposite angles of any quadrilateral figure described in a circle,[‡] are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD;

If in the first diagram, CE were produced to meet AB, the first case might be proved in a similar manner. The second case might also be proved by drawing from C to AB a straight line meeting it in a point H, and making with AC an angle equal to BAC. Then, by taking the difference between the equal angles EAC, ECA, and the equal ones HAC, HCA, we have EAB and ECH equal, and therefore $EBA=ECH$. But $EGB=HGC$; and therefore (I. 32, cor. 5) $BEC=BHC=2BAC$. The same proof, with some obvious variations, would be applicable in the first case.

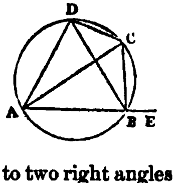
[*] There is evidently a third case, viz., when AB or AC passes through the centre: but though this case is not given in a separate form, its proof is contained in that of either of the others.

[†] The demonstration here adopted has the advantage of being applicable in all cases. In Simson's edition there are two cases, one in which the segment is greater than a semicircle, and the other in which it is not greater; the latter of which is wanting in the Greek.

[‡] That is, having all its angular points on the circumference, according to the second definition of the fourth book.

any two of its opposite angles are together equal to two right angles.

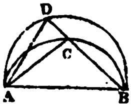
Join AC, BD. Then (III. 21) the angles CAB, CDB are equal, because they are in the same segment BADC; and the angles ACB, ADB are equal, because they are in the same segment ADCB: therefore (I. ax. 2) the whole angle ADC is equal to the angles CAB, ACB. To each of these equals add the angle ABC; therefore the angles ABC, CAB, BCA, are equal to the angles ABC, ADC: but (I. 32) ABC, CAB, BCA are equal to two right angles; therefore also the angles ABC, ADC are equal to two right angles. In the same manner, the angles BAD, BCD may be shown to be equal to two right angles. Therefore the opposite angles, &c.[*]



Cor. If any side AB be produced to E, the exterior angle CBE is equal to the interior and opposite angle ADC. For CBA, CBE are together equal to CBA, CDA, each pair being equal to two right angles. Take away CBA, and there remains CBE equal to CDA.

Prop. XXIII. Theor.—Upon the same straight line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, let the two similar segments of circles, viz. ACB, ADB, be upon the same side of the same straight line AB, not coinciding with one another: then (III. 10) because the circles ACB, ADB cut one another in the two points A, B, they cannot cut another in any other point. One of the segments must therefore fall within the other; let ACB fall within ADB, and draw the straight line BCD, cutting the arcs of the segments in C and D; join also CA, DA. Then, because the segment ACB is similar to the segment ADB, the angle ACB is equal (III. def. 8) to ADB, the exterior to the interior and remote, which (I. 16) is impossible. Therefore, upon the same straight line, &c.



Prop. XXIV. Theor.—Similar segments of circles upon equal bases are equal to one another, and have equal arcs.[†]

Let AEB, CFD be similar segments of circles upon the equal

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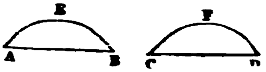
[*] In proving that the angles ABC, ADC are together equal to two right angles, we may either prove that the two parts of ADC are respectively equal to two angles of the triangle ABC, or that the two parts of ABC are respectively equal to two angles of the triangle ADC; and the proof is completed in each instance by adding the remaining angle of the triangle.

[†] The last clause, specifying that the segments have equal arcs, is improperly omitted in the Greek, and by Dr. Simson, and others.

straight lines or bases, AB, CD: the segments are equal; and likewise the arcs AEB, CFD are equal.

For, if the segment AEB be applied to the segment CFD, so that the point A may be on C, and the straight line AB on CD, the point B will coincide with D, because AB is equal to CD: therefore, the straight line AB coinciding with CD, the segment AEB must coincide (III. 23) with the segment CFD, and is therefore equal (I. ax. 8) to it: and the arcs AEB, CFD are equal, because they coincide. Therefore similar segments, &c.



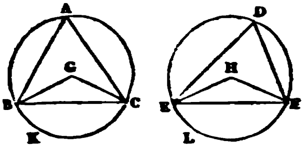
Prop XXV. Prob.—A segment of a circle being given, to complete the circle of which it is a segment.

Assume three points in the arc of the segment, and find (III. 1. schol.) the centre of the circle. From that centre, at the distance between it and any point in the arc describe a circle, and it will evidently be the one required.*

Prop. XXVI. Theor.—In equal circles, or in the same circle,† equal angles stand upon equal arcs, whether they are at the centres, or the circumferences.

Let ABC, DEF be equal circles, having the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences: the arc BKC is equal to the arc ELF.

Join BC, EF; and because the circles ABC, DEF are equal, their radii are equal: therefore the two sides BG, GC are equal to the two, EH, HF; and (hyp.) the angles G and H are equal: therefore (I. 4) the base BC is equal to the base EF. Then, because the angles A and D are equal, the segment BAC is similar (III. def. 8) to the segment EDF; and they are upon equal straight lines BC, EF; but (III. 24) similar segments of circles upon equal straight lines have equal arcs: therefore the arc BAC is equal to the arc EDF. But the whole circumference ABC is equal to the whole DEF, because the circles are equal: therefore the remaining arc BKC is equal (I. ax. 3) to the remaining arc ELF. Wherefore, in equal circles, &c.



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* This easy solution is substituted for the long one given by Euclid.

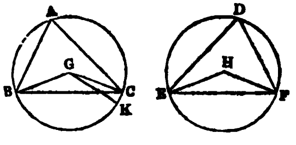
† In this proposition and some of the following, the words, "or in the same circle," are introduced in this edition; as what is proved in these propositions must evidently hold respecting the *same circle*, as well as equal ones: and they are often thus applied by Euclid himself.

Cor. Hence, in a circle the arcs intercepted between parallel chords are equal. For, if a straight line be drawn transversely, joining two extremities of the chords, it will (I. 29) make equal angles with the chords; and therefore the arcs on which these stand are equal.[*]

PROP. XXVII. THEOR.—In equal circles, or in the same circle, the angles which stand upon equal arcs are equal to one another, whether they are at the centres, or the circumferences.

In the equal circles ABC, DEF, let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences, stand upon the equal arcs BC, EF: the angle BGC is equal to EHF, and BAC to EDF.

If the angle BGC be equal to EHF, it is manifest (III. 20) that BAC is also equal to EDF. But, if BGC, EHF be not equal, one of them is the greater; let BGC, if possible, be the greater, and (I. 23) make BGK equal to EHF. Then (III. 26) equal angles stand upon equal arcs, when they are at the centres; therefore the arc BK is equal to the arc EF: but BC is equal (hyp.) to EF: therefore also BK is equal to BC, which (I. ax. 9) is impossible. Therefore the angle BGC is not unequal to EHF; that is, it is equal to it: and the angle A is half (III. 20) of BGC, and the angle D half of EHF: therefore (I. ax. 7) the angles A and D are equal. Wherefore, in equal circles, &c.



PROP. XXVIII. THEOR.—In equal circles, equal chords divide the circumferences into parts which are equal, each to each.[†]

Let ABC, DEF be equal circles, and BC, EF equal chords in them: the arc BAC is equal to EDF, and BGC to EHF.

If BC and EF be diameters, they bisect the circumferences; and those being equal, the parts of the one are equal to the parts of the other.

But if BC and DF be not diameters, take (III. 1) K, L, the centres of the circles, and join BK, KC, EL, LF. Then, because the circles are equal, BK, KC are equal to EL, LF; and the base BC is equal (hyp.) to the base EF; therefore (I. 8) the angle BKC is equal to ELF. But (III. 26) equal angles stand upon equal arcs, when they are at the centres: therefore the less arc BGC, in the one circle, is equal to the less arc EHF, in the other. But the whole circumference ABGC is equal to the whole DEHF; the remaining part, therefore, of the circumference, viz. the greater arc BAC, in the one circle, is equal to the remaining part, viz., the greater arc EDF in the other. Therefore in equal circles, &c.

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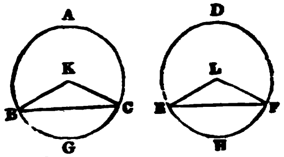
[*] It is plain from the next proposition, that the converse of this corollary is true.

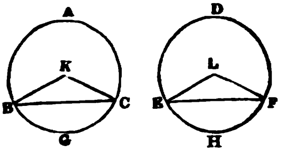
[†] Dr. Simson and most other editors omit, after the Greek, the case in which the chords are diameters. Some editors exclude this case in their enunciations, but give it in their proofs. The corresponding case in the next proposition is also omitted in most editions.



Prop. XXIX. Theor.—In equal circles, or in the same circle, equal arcs have equal chords.

Let ABC, DEF be equal circles, and let the arcs BGC, EHF, and consequently BAC, EDF be equal; and join BC, EF: the chords BC, EF are equal.

If BC and EF pass through the centres, they are equal, being diameters of equal circles. But if they do not pass through the centres, take (III. 1) K, L, the centres of the circles, and join BK, KC, EL, LF. Then, because the arc BGC is equal (hyp.) to the arc EHF, the angle BKC is equal (III. 27) to ELF. Also, because the circles ABC, DEF are equal, their radii are equal: therefore BK, KC are equal to EL, LF, and they contain equal angles: wherefore (I. 4) the base BC is equal to the base EF. Therefore, in equal circles, &c.

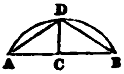


Prop. XXX. Prob.—To bisect a given arc of a circle.

Let ADB be a given arc; it is required to bisect it.

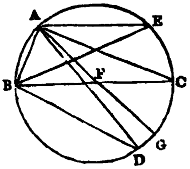
Draw AB, and (I. 10 and 11) bisect it in C, by the perpendicular CD; the arc ABD is bisected in the point D.

Join AD, DB. Then, because AC is equal to CB, CD common to the triangles ACD, BCD, and the angle ACD equal to BCD, each of them being a right angle; therefore (I. 4) AD is equal to BD. But, (III. 28) in the same circle, equal lines cut off equal arcs, the greater equal to the greater, and the less to the less; and AD, DB are each of them less than a semicircle, because (III. 1, cor.) DC, or DC produced, passes through the centre: wherefore the arc AD is equal to the arc DB: therefore the given arc is bisected in D: which was to be done.



PROP. XXXI. THEOR.—In a circle, (1) the angle in a semicircle is a right angle; (2) the angle in a segment greater than a semicircle is acute; and (3) the angle in a segment less than a semicircle is obtuse.

Let ABC be a circle, of which F is the centre, BC a diameter, and consequently BAC a semicircle; and let the segment BAD be greater, and BAE less than a semicircle: the angle BAC in the semicircle is a right angle; but the angle BAD in the segment greater than a semicircle is acute; and the angle BAE in the segment less than a semicircle is obtuse.

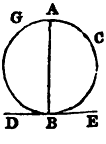


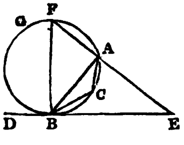
Draw AF and produce it to G. Then (III. 20) the angle BAG at the circumference, is half of BFG at the centre, both standing on the same arc BG; and for the same reason, GAC is half of GFC. Therefore the whole angle BAC is half of the angles BFG, GFC: and (I. 13) these are together equal to two right angles: therefore BAC is a right angle, and it is an angle in a semicircle. But (I. ax. 9) the angle BAD is less, and BAE greater than the right angle BAC: therefore an angle in a segment greater than a semicircle is acute, and an angle in a segment less than a semicircle is obtuse.

PROP. XXXII. THEOR.—If a straight line touch a circle, and from the point of contact a straight line be drawn dividing the circle into two segments; the angles made by this line with the tangent are equal to the angles which are in the alternate segments.

Let the straight line DE touch the circle BCG in the point B, and let the straight line BA be drawn dividing the circle into the segments AGB, ACB: the angle ABE is equal to any angle in the segment AGB, and the angle ABD to any angle in ACB.

If AB (fig. 1) be perpendicular to DE, it passes (III. 19) through the centre, and the segments being therefore semicircles, the angles in them are (III. 31) right angles, and consequently equal to those which AB makes with DE.*





But if BA (fig. 2) be not perpendicular to DE, draw BF perpendicular to it; join FA and produce it to E; join also CA, CB, C being any point in the arc

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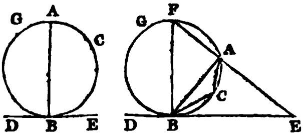
[*] This case is wanting in most editions of Euclid. In the second diagram, FA and DB will meet (I. ax. 12) if produced, the angles FBE, BFA together being evidently less than two right angles.

ACB. Then (III. 19) BF is a diameter, and (III. 31, and I. 13) the angles BAF, BAE are right angles. Therefore, in the triangles BAE, FBE, the angle E is common, and the angles BAE, FBE equal, being right angles; wherefore (I. 32, cor. 5) the remaining angle ABE is equal to the remaining angle F, which is an angle in the remote or alternate segment AGB.

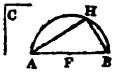


Again, the two angles ABD, ABE are equal (I. 13) to two right angles; and because ACBF is a quadrilateral in the circle, the opposite angles C and F are also equal (III. 22) to two right angles: therefore (I. ax. 1) the angles ABD, ABE are together equal to C and F. From these equals take away the angles ABE and F, which have been proved to be equal; then (I. ax. 3) the remaining angle ABD is equal to the remaining angle C, which is an angle in the remote segment ACB. If, therefore, a straight line, &c.*

Prop. XXXIII. Prob.—Upon a given straight line, to describe a segment of a circle, containing an angle equal to a given angle.

Let AB be the given straight line, and C the given angle; it is required to describe on AB a segment of a circle containing an angle equal to C.

First, if C be a right angle, bisect (I. 10) AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; therefore (III. 31) any angle AHB in the semicircle is equal to the right angle C.



But, if C be not a right angle, make (I. 23) the angle BAD equal to C, and (I. 11) from A draw AE perpendicular to AD; bisect (I. 10 and 11) AB by the perpendicular FG, and join GB. Then, because AF is equal to FB, FG common to the triangles

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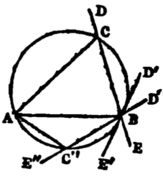
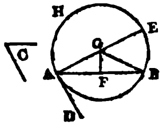
* This proposition, and the 21st and 22d, when properly considered, may be regarded as the same. To illustrate this, let AB be a fixed chord, and through B draw any other line DE cutting the circle in C; and join AC. Now, (III. 21) wherever C is taken in the arc ACB, the angle ACE is constantly of the same magnitude; and so also (I. 13) is the exterior angle ACD. If C be now taken as coinciding with B, the straight line DE will become the tangent D′BE′, AC will coincide with AB, and the angles ACE, ACD will become ABE′, ABD′. If again C take the position C″, the angles ACE, ACD will become AC″E″, AC″D″. Now, the equality of ABE′, ACE, and of ABD′, AC″D″ is what is proved in the 32d proposition: and from the equality of ACD and AC″D″ the 22d follows by the addition of ACB.



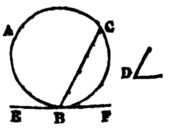
AFG, BFG, and the angle AFG equal to BFG; therefore (I. 4) AG is equal to GB; and the circle described from the centre G, at the distance GA will pass through the point B: let this be the circle AHB. Then, because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, AD (III. 16, cor.) touches the circle; and, (III. 32) because AB, drawn from the point of contact A cuts the circle, the angle DAB is equal to any angle in the alternate segment AHB: but DAB is equal to C; therefore also C is equal to any angle in the segment AHB: wherefore upon the given straight line AB the segment AHB is described, which contains an angle equal to C: which was to be done.*



PROP. XXXIV. PROB.—From a given circle to cut off a segment which shall contain an angle equal to a given angle.

Let ABC be a given circle, and D a given angle; it is required to cut a segment from ABC containing an angle equal to D.

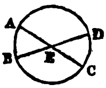
Draw (III. 17) the straight line EF touching the circle in B, and (I. 23) make the angle FBC equal to D. Then, because EF touches the circle, and BC is drawn from the point of contact B, the angle FBC is equal (III. 32) to any angle in the alternate segment BAC: but FBC is equal to D: therefore any angle in the segment BAC is equal to D: wherefore the segment BAC is cut off from the given circle, containing an angle equal to the given angle D: which was to be done.



PROP. XXXV. THEOR.—If two chords of a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

In the circle ABCD, let the two chords AC, BD cut one another in the point E: the rectangle AE.EC is equal to the rectangle BE.ED.

If AC, BD both pass through the centre, so that E is the centre; it is evident, that AE, EC, BE, ED, being (I. def. 30) all equal, the rectangle AE.EC is equal to the rectangle BE.ED.



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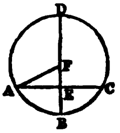
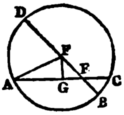
[*] It is evident, there may be two segments answering the conditions of the problem, one on each side of the given line. It is also plain, that when C is an acute angle, and the segment is to be above AB, G is above AB; but when obtuse, it is below it. It is likewise plain, that the angle BAE is the complement of the given angle C; that is, the difference between it and a right angle.

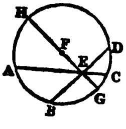
But let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, in the point E. Then, if BD be bisected in F, F is the centre. Join AF; and because BD, which passes through the centre, is perpendicular to AC, AE, EC are (III. 3) equal to one another. Now, because BD is divided equally in F, and unequally in E, the rectangle BE.ED, and the square of EF are equal (II. 5) to the square of FB; that is, (I. 46, cor. 2) to the square of FA. But (I. 47) the squares of AE, EF are equal to the square of FA; therefore the rectangle BE.ED, and the square of EF are equal to the squares of AE, EF. Take away the common square of EF, and the remaining rectangle BE.ED is equal to the remaining square of AE; that is, to the rectangle AE.EC.



Next, let BD pass through the centre, and cut AC, which does not pass through the centre, in E, but not at right angles. Then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and (I. 12) draw FG perpendicular to AC; therefore (III. 8) AG is equal to GC; wherefore (II. 5) the rectangle AE.EC and the square of EG are equal to the square of AG. To each of these equals add the square of GF: therefore the rectangle AE.EC and the squares of EG, GF are equal to the squares of AG, GF: but (I. 47) the squares of EG, GF are equal to the square of EF; and the squares of AG, GF are equal to the square of AF: therefore the rectangle AE.EC and the square of EF are equal to the square of AF; that is, (I. 46, cor. 2) to the square of FB. But (II. 5) the square of FB is equal to the rectangle BE.ED, together with the square of EF; therefore the rectangle AE.EC and the square of EF are equal to the rectangle BE.ED and the square of EF; take away the common square of EF, and the remaining rectangle AE.EC is equal to the remaining rectangle BE.ED.*



Lastly, let neither of the lines AC, BD pass through the centre. Take the centre F, and through E draw the diameter GEFH. Then, the rectangle AE.EC is equal, as has been shown, to the rectangle GE.EH; and, for the same reason, the rectangle BE.ED is equal to the same rectangle GE.EH: therefore (I. ax. 1) the rectangle AE.EC is equal to the rectangle BE.ED. If, therefore, two chords of a circle, &c.



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* The second and third cases may be thus demonstrated in one:

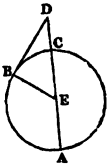
Join FC. Then the rectangle AE.EC is equal (II. 5, cor. 5) to the difference of the squares of AF and FE, or of DF and FE, or (II. 5. cor. 1) to the rectangle DE.EB.

Proportion, however, affords much the easiest method of demonstrating both this proposition and the following. See the Notes at the end of the volume.

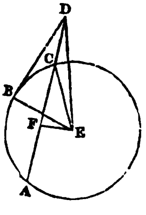
Prop. XXXVI. Theor.—**If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, is equal to the square of the line which touches it.**

Let D be any point without the circle ABC, and DCA, DB two straight lines drawn from it, of which DCA cuts the circle in C and A, and DB touches it in B: the rectangle AD.DC is equal to the square of DB.

Either DCA passes through the centre, or it does not: first, let it pass through the centre E, and join EB. Therefore (III. 18) the angle EBD is a right angle: and (II. 6) because the straight line AC is bisected in E, and produced to D, the rectangle AD.DC and the square of EC are together equal to the square of ED; and CE is equal to EB: therefore the rectangle AD.DC and the square of EB are equal to the square of ED. But (I. 47) the square of ED is equal to the squares of EB, BD, because EBD is a right angle: therefore the rectangle AD.DC and the square of EB are equal to the squares of EB, BD; take away the common square of EB; and the remaining rectangle AD.DC is equal to the square of the tangent DB.



But if DCA do not pass through the centre, take (III. 1) the centre E, and draw (I. 12) EF perpendicular to AC, and join EB, EC, ED. Then, because the straight line EF, which passes through the centre, is perpendicular to the chord AC, AF is equal (III. 3) to FC. And (II. 6) because AC is bisectcd in F, and produced to D, the rectangle AD.DC and the square of FC are equal to the square of FD. To each of these equals add the square of FE: therefore the rectangle AD.DC, and the squares of CF, FE are equal to the squares of DF, FE: but (I. 47) the square of ED is equal to the squares of DF, FE, because EFD is a right angle; and the square of EC, or (I. 46, cor. 2) of EB, is equal to the squares of CF, FE: therefore the rectangle AD.DC and the square of EB are equal to the square of ED. But (I. 47) the squares of EB, BD are equal to the square of ED, because EBD is a right angle: therefore the rectangle AD.DC and the square of EB are equal to the squares of EB, BD. Take away the common square of EB: therefore the remaining rectangle AD.DC is equal to the square of DB: wherefore, if from any point, &c.*



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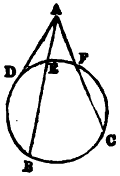
* The second case may be demonstrated more briefly thus: Join AE. Then the rectangle AD.DC is equal (II. 5, cor. 5) to the difference of the squares of ED and EC, or of ED and EB, or (I. 47) to the square of DB.

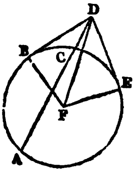
Cor. If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA.AE to the rectangle CA.AF: for each rectangle is equal to the square of the tangent AD.



Prop. XXXVII. Theor.—**If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets the circle touches it.**

If from a point without the circle ABC, two straight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; and if the rectangle AD.DC be equal to the square of DB; DB touches the circle.

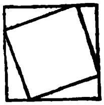
Draw (III. 17) the straight line DE touching the circle ABC; find (III. 1) the centre F; and join FE, FB, FD. Then (III. 18) FED is a right angle: and (III. 36) because DE touches the circle ABC, and DCA cuts it, the rectangle AD.DC is equal to the square of DE. But (hyp.) the rectangle AD.DC, is equal to the square of DB: therefore the square of DE is equal to the square of DB; and the straight line DE equal (I. 46, cor. 3) to the straight line DB. But FE is equal to FB, and the base FD is common to the two triangles DEF, DBF; therefore (I. 8) the angle DEF is equal to DBF; but DEF is a right angle; therefore also DBF is a right angle: and FB is a part of a diameter, and the straight line which is drawn at right angles to a diameter, from its extremity, touches (III. 16, cor.) the circle: therefore DB touches the circle ABC. Wherefore, if from a point, &c.



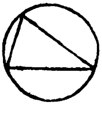
BOOK IV.*

DEFINITIONS.

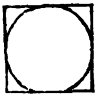
1. When the sides of one rectilineal figure pass through the angular points of another, the figures not coinciding with one another, the interior figure is said to be *inscribed* in the exterior, and the exterior to be *circumscribed*, or *described*, about the interior one.



2. When all the angular points of a rectilineal figure are upon the circumference of a circle, the rectilineal figure is said to be *inscribed* in the circle, and the circle to be *circumscribed*, or *described*, about the rectilineal figure.



3. When each side of a rectilineal figure touches a circle, the rectilineal figure is said to be *circumscribed*, or *described*, about the circle, and the circle to be *inscribed* in the rectilineal figure.

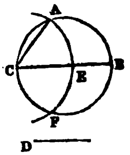


PROPOSITIONS.

Prop. I. Prob.—In a given circle to place a chord equal to a given straight line, not greater than the diameter.

Let ABC be a given circle, and D a given straight line not greater than the diameter of the circle: it is required to place in ABC a chord equal to D.

Draw BC a diameter of the circle ABC; then, if BC be equal to D, the thing required is done; for in the circle ABC a chord BC is placed equal to D. But if it be not, BC is greater than D; make CE equal (I. 3) to D, and from the centre C, at the distance CE, describe (I. post. 3) the circle AEF, and join CA. Then, because C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE: therefore CA is equal to D: wherefore in the circle ABC, a chord is placed equal to the given straight line D: which was to be done.



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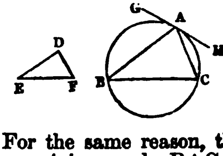
* This book consists of propositions, chiefly respecting the rectilineal figures described in circles or about them.

Prop. II. Prob.—In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be a given circle, and DEF a given triangle; it is required to inscribe in ABC a triangle equiangular to DEF.

Draw the straight line GAH (III. 17) touching the circle in any point A, and make (I. 23) the angle HAC equal to E, and GAB equal to F, and join BC: ABC is the triangle required.

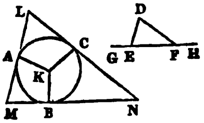


For, since GAH touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal (III. 32) to the angle B in the alternate segment; but HAC is equal (const.) to E: therefore also B is equal to E. For the same reason, the angle C is equal to F: therefore the remaining angle BAC is equal (I. 32, cor. 5) to the remaining angle D: wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed (IV. def. 2) in the circle ABC: which was to be done.

Prop. III. Prob.—About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be a given circle, and DEF a given triangle; it is required to describe a triangle about ABC, equiangular to DEF.

Produce EF both ways to G, H, and find (III. 1) K the centre of the circle; from it draw any radius KB; and make (I. 23) the angle BKA equal to DEG, and BKC equal to DFH; through the points A, B, C, draw (III. 17) the tangents LAM, MBN, NCL: LMN is the required triangle.*



For, since LM, MN, NL touch the circle ABC in the points A, B, C, to which KA, KB, KC are drawn from the centre, the angles at the points A, B, C, are (III. 18) right angles: and because the four angles of the quadrilateral figure AMBK are equal (I. 32, cor. 1) to four right angles; and that two of them, KAM, KBM, are right angles, the other two AKB, AMB are equal to two right angles. But (I. 13) the angles DEG, DEF are likewise equal to two right angles: therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which (const.) AKB is equal to DEG: wherefore the remaining angles AMB, DEF are equal. In like manner, the angle LNM may be demonstrated to be equal to DFE; and therefore the remaining angle MLN is equal (I. 32, cor. 5) to the remaining angle EDF: wherefore the triangle LMN is equiangular to DEF: and (IV. def. 3) it is described about the circle ABC: which was to be done.

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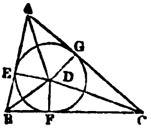
* That ML and NL must meet, might be shown by joining AC; for the angles CAL, ACL being less than the right angles at A and C, AL and CL (I. ax. 12) must meet. In a similar manner it might be proved that LM and NM, and also LN and MN must meet.

PROP. IV. PROB.—To inscribe a circle in a given triangle.

Let ABC be a given triangle; it is required to inscribe a circle in it.

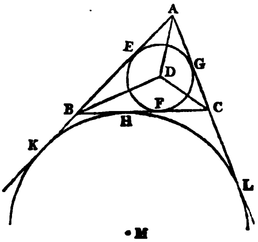
Bisect (I. 9) the angles ABC, BCA by the straight lines BD, CD, meeting one another in the point D; and from D draw (I. 12) DE, DF, DG, perpendiculars to AB, BC, CA. Then, because the angle EBD is equal (const.) to FBD, and that the right angles BED, BFD are equal, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, each to each, and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore (I. 26) DE is equal to DF. In the same manner it would be shown, that, in the triangles DGC, DFC, DG is equal to DF: therefore the three straight lines DE, DF, DG are equal to one another, and the circle described from the centre D, at the distance of any of them, will pass through the extremities of the other two; and (III. 16, cor.) it will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles. Therefore the straight lines AB, BC, CA touch the circle, and the circle EFG is inscribed (IV. def. 3) in the triangle ABC; which was to be done.



Schol. This proposition is a case of the general problem, *to describe a circle touching three given straight lines which do not pass through the same point, and which are not all parallel to one another.*

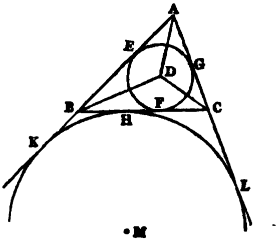
If two of the lines be parallel, there may evidently be two equal circles, one on each side of the line falling on the parallels, each of which will touch the three given lines: and their centres will be the intersections of the lines bisecting the angles made by the parallels with the third line.

If the lines form a triangle by their intersections, there will be four circles touching them; one, literally, inscribed, and the others each touching one side externally, and the other two produced. The centres of the external circles will be the intersections of the lines bisecting two exterior angles; and the line bisecting the remote interior angle will pass through the same point. The method of proof respecting the external circles is almost the same as that of the foregoing proposition.



Cor. 1. Join AD. Then (I. 47, cor. 5) the angles EAD, GAD are equal. Hence the three straight lines bisecting the three angles of a triangle meet in the same point, and that point is the centre of the inscribed circle.

Cor. 2. AE is equal (III. 17, schol.) to AG, BE to BF, and CF to CG. Hence the sum of any side AB, and the remote segment, CF or CG, of either of the other sides, is equal to half the perimeter: and consequently either of these segments, CF or CG, is equal to what remains, when the third side AB is taken from half the perimeter.



Cor. 3. If the circle KHL touch BC, a side of the triangle ABC, in H, and the other sides produced in K and L, AK or AL is equal to half the perimeter. For (III. 17, schol.) BK is equal to BH, CL to CH, and AK to AL: therefore AK and AL are equal to the perimeter, and each of them to half the perimeter.

Cor. 4. Since (cor. 2) AB and FC are equal to half the perimeter, and AK equal (cor. 3) to the same; if AB be taken away there remains BK or BH equal to CF or CG.

Cor. 5. If C be a right angle, CG or CF is evidently equal to DG or DF, the radius of the inscribed circle. Hence, in a right-angled triangle the diameter of the inscribed circle is equal to the excess of the sum of the legs above the hypotenuse. It would appear, in like manner, that the diameter of the circle touching the hypotenuse externally, and the legs produced, is equal to the perimeter.

Cor. 6. Since the triangles ADB, BDC, and CDA are respectively equal (I. 41) to the rectangles under the radius of the inscribed circle, and the halves of AB, BC, and CA, it follows (II. 1) that the area of ABC is equal to the rectangle under the radius and half the perimeter. Hence, if the sides be given in numbers, the length of the radius may be computed by calculating (II. 13, note) the area, and dividing it by half the sum of the sides, or its double by their sum.

Cor. 7. If straight lines be drawn from M, the centre of one of the external circles, to A, B, and C, the triangles AMB, AMC, and BMC, are respectively equal (I. 41) to the rectangles under the radius of that circle and the halves of AB, AC, and BC. Hence, if the last of these be taken from the sum of the others, there remains the triangle ABC equal to the rectangle under the radius, and the excess of half the sum of AB and AC above the half of BC; or, which is the same, to the rectangle under the radius, and the excess[*] of half the perimeter above BC. The radius, therefore, of any of the external circles may be computed by dividing the area by the excess of half the perimeter above the side which the circle touches externally.

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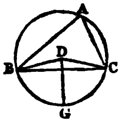
[*] This excess (cor. 2) is AE or AG; and hence, if AM and EM were joined, the triangle AEM would be equal to half the triangle ABC.

PROP. V. PROB.—To describe a circle about a given triangle.

Let the given triangle be ABC; it is required to describe a circle about it.



Find (III. 1, schol.) D the centre of the circle passing through A, B, C, the three angular points of the triangle, and the circle described from D as centre, at the distance of any of the angular points A, B, C, is evidently the circle required.

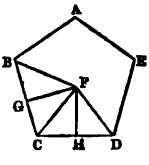
Cor. 1. It is evident (from this proposition, and from III. 1, schol.), that the three straight lines bisecting the three sides and perpendicular to them, meet in the same point, and that the point in which they meet is the centre of the circumscribed circle.

Cor. 2. Through the centre D draw DG perpendicular to BC: the angle BDG or CDG is equal to BAC. For (I. 4) the angles BDG, CDG are equal, as are also (III. 26) the arcs BG, CG. Therefore (III. 27) the straight line joining AG bisects the angle BAC: and (III. 20) each of the angles BDG, CDG is double of either half of BAC.

PROP. A. THEOR.*—Any equilateral and equiangular rectilineal figure may have one circle described about it, and another in it; and the same point is the centre of both circles.

Let ABCDE be a rectilineal figure, having all its sides equal, and all its angles equal: one circle may be described about it, and another in it, and the same point is the centre of both.

Bisect the angles BCD, CDE by the lines CF and DF: these (I. ax. 12) will meet, if produced: let them be produced and meet in F, and join FB. Then, since the angles FCD, FDC are equal, being the halves of equal angles, FC is equal (I. 6) to FD. Also, in the triangles BCF, DCF, the side BC is equal (hyp.) to DC, CF is common, and (const.) the angle BCF is equal to DCF: therefore (I. 4) BF is equal to DF, and consequently to CF. The angle CBF also is equal to CDF, and is therefore the half of ABC. In the same manner, the straight lines drawn from F to the other angles A and E may be proved to be equal to BF, CF, or DF: and therefore a circle described from F as centre, at a distance equal to any of these lines, will pass through all the angular points, and (IV. def. 2) will be circumscribed about the rectilineal figure ABCDE.



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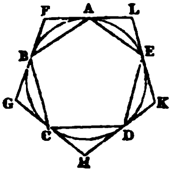
* This proposition and the next are introduced as lemmas, for the purpose of generalizing and abbreviating much of what follows in this book.

From F draw FG perpendicular to BC, and FH to CD. Then, since BC and CD are equal chords of the circumscribed circle, they are (III. 14) equally distant from the centre: that is, FG is equal to FH: and a circle described from F as centre at the distance of either of these, will pass through G and H; and it will touch (III. 16) BC and CD, because the angles at G and H are right angles: and in the same manner it might be shown, that the same circle would touch the other sides of the figure: it would therefore (IV. def. 3) be inscribed in the rectilineal figure ABCDE: wherefore any equilateral and equiangular rectilineal figure, &c.

Cor. BC is bisected (III. 3) in G, and CD in H; and (const.) FG is perpendicular to BC, and FH to CD. Hence the centre of the inscribed or circumscribed circle may be found by assuming any three of the angular points, and following the method pointed out in the scholium to the first proposition of the third book; or by bisecting two adjacent sides by perpendiculars: and the radius of the inscribed circle is the perpendicular drawn from F to any of the sides, while that of the circumscribed circle, is the straight line drawn from the same point to any of the angles.

Prop. B. Theor.—If any equilateral and equiangular rectilineal figure be inscribed in a circle, tangents to the circle, drawn through the angular points, will form an equilateral and equiangular figure of the same number of sides, described about the circle.

Let ABCDE be an equilateral and equiangular rectilineal figure described in the circle ACE: if tangents be drawn to the circle through the points A, B, C, D, E, meeting in the points F, G, H, K, L, the figure FGHKL is also an equilateral and equiangular figure of the same number of sides as the figure ABCDE.



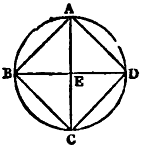
For, since GH touches the circle in C, and CD cuts it, the angle DCH is equal (III. 32) to any angle at the circumference standing on the arc CD; and because HK touches the circle in D, and CD cuts it, the angle CDH is also equal to any angle at the circumference standing on the same arc CD: therefore the angles DCH, CDH are equal to one another. For the same reason EDK, DEK are each equal to any angle at the circumference standing on the arc DE: but (III. 28) the arcs AB, BC, CD, &c. are all equal, because (hyp.) their chords are equal; and (III. 27) angles at the circumference standing on these arcs are equal; and therefore the angles DCH, CDH, EDK, DEK, LEA, &c. are all equal. Hence (I. 6) CH is equal to HD, DK to KE, &c. Also, in the triangles CHD, DKE, there are two angles of the one equal to two angles of the other, each to each, and (hyp.) the sides CD, DE are equal: therefore (I. 26) the sides CH, HD are equal to DK, KE, each to each, and the angle H to K. In the same manner, it would be shown that the angle K is equal to

L, L to F, &c.; and therefore the figure FGHKL is equiangular. It is also equilateral; for its sides are the doubles of the equal lines FB, GC, &c. Therefore, if any equilateral, &c.

PROP. VI. PROB.—To inscribe a square in a given circle.

Let ABCD be the given circle; it is required to inscribe a square in it.

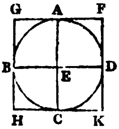
Draw two diameters AC, BD at right angles to one another; and join AB, BC, CD, DA; ABCD is the square required.



Because (I. def. 30) BE is equal to ED, and EA is common, and at right angles to BD; BA is equal (I. 4) to AD. Now, the angles CBA, BAD being angles in semicircles, are (III. 31) right angles; and therefore (I. 28, part 2) AD is parallel to BC. For the same reason AB is parallel to DC; and therefore (I. def. 24) ABCD is a parallelogram. But it has been shown that BAD is a right angle, and that the sides BA, AD containing it are equal: ABCD is therefore (I. def. 27) a square, and (IV. def. 2) it is inscribed in the circle ABCD: which was to be done.

PROP. VII. PROB.—To describe a square about a given circle.

Let ABCD be a given circle; it is required to describe a square about it.

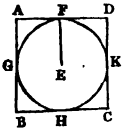


Draw two diameters, AC, BD, at right angles to one another, and (III. 17) through the points A, B, C, D draw FG, GH, HK, KF, touching the circle: FGHK is the square required.

For, by the preceding proposition, A, B, C, D are the angular points of the inscribed square; and therefore (IV. B) the figure FGHK formed by the tangents drawn through these points, is a square described about the circle: which was to be done.

PROP. VIII. PROB.—To incribe a circle in a given square.

Let ABCD be the given square; it is required to inscribe a circle in it.



Find (IV. A, cor.) the centre E of the inscribed circle, and draw (I. 12) EF perpendicular to AD: from E as centre, with EF as radius, describe the circle FGHK: it is (IV. A, cor.) the circle required.*

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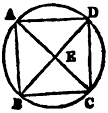
* In practice, the construction is effected most easily, both in this proposition and the following, by drawing the diagonals, as their point of intersection is evidently the centre.

Prop. IX. Prob.—To describe a circle about a given square.

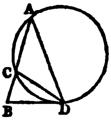


Let ABCD be the given square; it is required to describe a circle about it.

Find, as in the preceding proposition, the centre E; and from that centre, at the distance EA, describe the circle ABCD: it is (IV. A, cor.) the circle required.

Prop. X. Prob.—To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and divide it (II. 11) in C, so that the rectangle AB.BC may be equal to the square of AC; describe (I. 22) the triangle ABD, having AD equal to AB, and BD to AC: ABD is a triangle, such as is required; that is, each of the angles ABD, ADB is double of A.



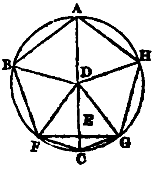
Join CD, and (IV. 5) about the triangle ADC describe the circle ACD. Then (I. 5) the angles ABD, ADB are equal. But (const.) the rectangle AB.BC is equal to the square of AC, or (const.) to the square of BD: therefore since the rectangle under AB, the whole line which cuts the circle, and BC, the part of it without the circle, is equal to the square of BD, which meets the circle, BD (III. 37) touches the circle in D; and therefore (III. 32) the angle BDC contained by BD which touches the circle, and CD which cuts it, is equal to A, the angle in the alternate segment. To each of these equals add the angle ADC; then (I. ax. 2) the whole angle ADB, or its equal B, is equal to the two angles A and ADC: but the exterior angle BCD is also equal (I. 32) to the two angles A and ADC: therefore (I. ax. 1) the angles B and BCD are equal, and therefore (I. 6) the sides BD, CD are equal. Hence also the sides AC, DC are equal, each being equal to BD; and therefore (I. 5) the angles A and ADC are equal. But B has been proved to be equal to A and ADC; and therefore, since these are equal to one another, B and its equal ADB are each double of A; wherefore a triangle ABD has been described, which has each of the angles at its base BD double of the third angle A: which was to be done.

Cor. Since (I. 32) the three angles of a triangle are together equal to two right angles, it is plain, that the angle A is one fifth, and each of the angles at the base BD, two fifths of two right angles. Hence also the angle A is one tenth of four right angles.

Prop. XI. Prob.—To inscribe a regular pentagon, that is, an equilateral and equiangular pentagon, in a given circle.

Let ABG be a given circle; it is required to inscribe in it a regular pentagon.

Draw any radius DC, and (II. 11) divide it in E, so that the rectangle DC.CE may be equal to the square of DE: inscribe (IV. 1) the chords CF, CG, each equal to DE, and join FG: FG is a side of the required pentagon. For (IV. 10, cor.) each of the angles FDC, GDC is one tenth of four right angles; and therefore FDG is two tenths, or one fifth of four right angles, that is, of all the angles (I. 13, cor. 2) made by any number of lines meeting in D. Make therefore (I. 23) the angles GDH, HDA, ADB, each equal to FDG, and BDF will also be equal to FDG. Join GH, HA, AB, BF. Now (III. 26) equal angles at the centre of a circle stand on equal arcs, and (III. 29) equal arcs have equal chords; therefore the chords FG, GH, HA, AB, BF are all equal, and the pentagon ABFGH is equilateral. It is also equiangular; for each of its angles stands on an arc which is treble of the arc FG; and (III. 27) the angles which stand on equal arcs are equal to one another, whether they be at the centre or the circumference. In the given circle, therefore, a regular pentagon has been described; which was to be done.*



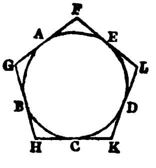
Cor. 1. It is evident from this demonstration, that any equilateral polygon described in a circle is also equiangular.

Cor. 2. It is evident, also, that FC or CG is a side of a regular decagon inscribed in the circle; and thus we have the means of describing that figure in a given circle.

PROP. XII. PROB.—To describe a regular pentagon about a given circle.

Let ABCDE be a given circle; it is required to describe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angular points of a pentagon, inscribed in the circle, by the last proposition, be A, B, C, D, E; and through these points draw (III. 17) the straight lines FG, GH, HK, KI, IF, touching the circle: the figure FGHKI is (IV. B) a regular pentagon described about the circle: which was to be done.



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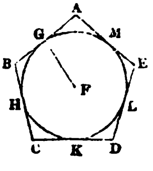
* In the construction given by Euclid, a triangle is described as in the last proposition; then a triangle equiangular to it is described (IV. 2) in the circle: the angular points of this triangle are three of the angular points of the pentagon, and the remaining points are found by bisecting the angles at the base of the inscribed triangle. The construction here given is considerably easier in practice. It may be remarked, that after the chord FG is drawn, the construction of the pentagon is completed simply by inscribing the chords, GH, HA, &c. each equal to FG.

PROP. XIII. PROB.—To inscribe a circle in a given regular pentagon.

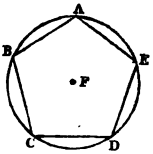
Let ABCDE be a given regular pentagon; it is required to inscribe a circle in it.



Find (III. 1. schol.) F the centre of the circle whose circumference would pass through the three points A, B, C; this (IV. A, cor.) is the centre of the circle inscribed in the pentagon. Draw FG perpendicular to AB, and from F as centre, with FG as radius, describe the circle GHKLM: this (by the same corollary) is the circle described in the given pentagon: which was to be done.

PROP. XIV. PROB.—To describe a circle about a given regular pentagon.

Let ABCDE be the given regular pentagon; it is required to describe a circle about it.

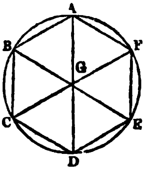


Find the centre F, as in the preceding proposition; and from that centre, at the distance FA, describe the circle ABCDE: this (IV. A, cor.) is a circle described about the given pentagon: which was to be done.

PROP. XV. PROB.—To inscribe a regular hexagon in a given circle.

Let ABCDEF be a given circle; it is required to inscribe a regular hexagon in it.

Find G the centre of a given circle, and draw any radius AG: inscribe (IV. 1) the chord AB equal to AG, and join GB. Then, (I. 32, and I. 5, cor.) since the triangle AGB is equilateral, the angle AGB is a third of two right angles, or a sixth of four. But (I. 13) cor. 2) the angles that can be made about the point G, are equal to four right angles: therefore, if the angles BGC, CGD, DGE, EGF be each made equal to AGB, AGF will also be equal to AGB. The angles at G, therefore, are all equal; and (III. 26) equal angles at the centre stand on equal arcs, and (III. 29) equal arcs have equal chords; therefore the figure ABCDEF is equilateral. It is also inscribed in the circle; and, therefore (IV. 11, cor. 1) it is equiangular: wherefore in the given circle a regular hexagon has been described: which was to be done.



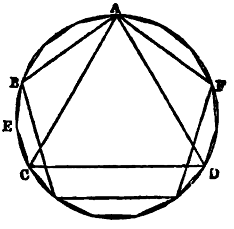
Cor. From this it is manifest, that each side of the hexagon is equal to the radius of the circle.

Schol. If, through the points A, B, C, E, F, there be drawn straight lines touching the circle, a regular hexagon will (IV. B) be described about it, and likewise a circle may be inscribed in a given regular hexagon, and also one circumscribed about it by a method similar to that used for the pentagon.

PROP. XVI. PROB.—To inscribe a regular quindecagon in a given circle.

Let ABCD be a given circle; it is required to inscribe in it an equilateral and equiangular quindecagon.

Let AC be a side of an equilateral triangle inscribed (IV. 2) in the circle, and AB a side of a regular pentagon inscribed (IV. 11) in the same; therefore, of such equal parts as the whole circumference ABCDF contains fifteen, the arc ABC, being the third part of the whole, contains five; and the arc AB, which is the fifth part of the whole, contains three; therefore BC, their difference, contains two of the same parts. Bisect, therefore (III. 30), BC in E; then BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: and therefore, if the straight lines BE, EC be drawn, and chords each equal to one of them be placed around in the whole circle, a regular quindecagon will be inscribed in it: which was to be done.



In the same manner, also as was done in case of the pentagon, if, through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, a regular quindecagon will be described about it: and likewise, as in the pentagon, a circle may be inscribed in a given regular quindecagon, and also one circumscribed about it.

Schol. Any regular polygon being inscribed in a circle, if the arcs cut off by its sides be bisected, and the points of bisection be joined with the nearest angular points, a polygon of double the number of sides is obtained: and, by repetitions of the process, as many regular polygons as we please, may be inscribed, each having twice as many sides as the one immediately preceding it.*

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* The polygons discussed in this book and those which may be derived from them by the process pointed out in the preceding scholium, are the only ones which geometers were able, till lately, to describe by elementary geometry, that is, by means of the straight line and circle. In 1801, however, M. Gauss of Gottingen, in a work entitled *Disquisitiones Arithmeticæ*, showed that in a circle, by elementary geometry, every regular polygon may be inscribed, the number of whose sides is a power of 2 increased by unity, and is a prime number, that is, a number which is not produced by the multiplication of any two whole numbers; such as 17, which is the fourth power of 2 increased by one. Such also are polygons of 257, and 65537 sides. The investigation of the means of describing these, is too complex and difficult to be given here.

BOOK V.[*]

DEFINITIONS.

1. A LESS number or magnitude is said to *measure* a greater, or to be a *measure*, a *part*, or a *submultiple* of the greater, when the less is contained a certain number of times, exactly, in the greater: and

2. The greater is said to be a *multiple* of the less.[†]

3. Magnitudes which can be compared in respect of quantity, that is, which are either equal to one another, or unequal, are said to be of the *same kind*.[‡]

4. If there be two magnitudes of the same kind, the relation which one of them bears to the other in respect of quantity, is called its *ratio* to the other.

The first term, or magnitude, is called the *antecedent* of the ratio, and the second the *consequent*.[§]

5. If there be four magnitudes, and if any like multiples whatever be taken of the first and third, and any whatever of the

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[*] This book contains the general theory of proportion, and the principles necessary for establishing it.

[†] Thus, a line of 15 inches is a multiple of a line of 5 inches, containing it exactly three times; and a line of 5 inches is a measure, a part, or a submultiple of a line of 15 inches. On the contrary, a line of 14 inches is not a multiple of a line of 5 inches, nor a line of 5 inches a submultiple of one of 14 inches. It may be remarked, that the word *part*, in this restricted use of it, is improperly applied, and should be laid aside, 5 being, in the true sense of the word, a *part* of 14 as well as of 15. To obviate this, the expression *aliquot part* is sometimes employed. The term *submultiple*, however, seems to be preferable to any other, from its relation to the corresponding term *multiple*.

A number or magnitude which is a multiple of two or more others, is said to be a *common multiple* of them. Thus, 12 is a common multiple of 2, 3, 4, and 6. In like manner, a number, or magnitude, which is a measure of two or more others, is said to be a *common measure* of them. Thus, 4 is a common measure of 16, 20, and 36. Numbers or magnitudes which have a common measure, are said to be *commensurable* to one another; while those which have no common measure, are *incommensurable*.

[‡] Thus, lines, whether straight or curved, are magnitudes of the same kind, or are *homogeneous*, since they may be equal or unequal. In like manner, surfaces, solids, and angles form three other classes of homogeneous magnitudes. On the contrary, lines and surfaces, lines and angles, surfaces and solids, &c. are heterogeneous. Thus, it is obviously improper to say, that the side and area of a square are equal to one another, or are unequal. So likewise we cannot say that the area and one of the angles of a triangle are equal to each other, or are unequal; and they are therefore heterogeneous.

[§] Thus, a line of 10 inches has a certain relation in respect of magnitude to a line of 5 inches, being double of it. This relation is called the *ratio* of the first to the second; and the line of 10 inches is called the *antecedent* of the ratio, and the other its *consequent*.

second and fourth; and if, according as the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third is also greater than the multiple of the fourth, equal to it, or less: then the first of the magnitudes is said to have to the second the *same ratio* that the third has to the fourth.*

6. Magnitudes which have the same ratio are called *proportionals:* and equality or identity of ratios constitutes *proportion* or *analogy.*

When magnitudes are proportionals, the relation is expressed briefly by saying, that the first *is to* the second, *as* the third *to* the fourth, the fifth *to* the sixth, and so on.†

7. When of the multiples of four magnitudes, taken as in the fifth definition, the multiple of the first is greater than that of the second, but the multiple of the third is not greater than that of the fourth; then the first is said to have to the second a *greater ratio* than the third has to the fourth; and, on the contrary, the third is said to have to the fourth a *less ratio* than the first has to the second.

8. When there are three‡ or more magnitudes of the same kind, such that the ratios of the first to the second, of the second to the third, and so on, whatever may be their number, are all equal; the magnitudes are said to be *continual proportionals.*§

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* It is plain that the first and second magnitudes must be of the same kind; and that the third and fourth must also be of the same kind compared with one another, though they may differ in kind from the first and second. A similar remark is applicable in reference to the next two definitions.

The properties of proportionals may in many instances be familiarly illustrated by means of numbers. Thus, to illustrate this definition, let us take the numbers 10, 4, 15, and 6, and it will be seen that the ratio of the first to the second is the same as the ratio of the third to the fourth. For, if the first and third be multiplied by 2, and the second and fourth by 5, the multiples of the first and second are each 20, and those of the others each 30: but if, instead of 2 and 5, the multipliers 3 and 8 be employed, the multiples of the first and third are respectively less than those of the second and fourth:—a relation which is reversed, if the multipliers 4 and 9 be used. The operations will stand thus:—

10, 4, 15, 6.	10, 4, 15, 6.	10, 4, 15, 6.
2, 5, 2, 5.	3, 8, 3, 8.	4, 9, 4, 9.
20, 20, 30, 30.	30, 32, 45, 48.	40, 36, 60, 54.

In this definition, it must be understood that the relations hold, not merely for particular multiples, but for *all multiples whatever.* This will be exemplified by the numbers 7, 4, 9, 6, which would appear to exhibit equal ratios, if the first and third were multiplied by 3, and the second and fourth by 4; but not if the first and third be multiplied by 5 and the others by 8: and by the 7th definition it will appear that 7 has to 4 a greater ratio than 9 has to 6. This consideration must be kept in view in every elementary application of the theory of proportion.

† Thus, 6, 9, 8, 12, 10, 15, are proportionals, the ratios of 6 to 9, of 8 to 12, and of 10 to 15 being all equal; and we briefly express this relation by saying, that 6 is to 9, as 8 to 12, and as 10 to 15. We also usually write proportionals in succession with two dots between the terms of the ratios, and four between the others: thus, 6 : 9 :: 8 : 12 :: 10 : 15.

‡ It is plain that proportion, being the equality of ratios, consists in three terms at least: as with fewer terms there could not be two or more ratios.

§ Hence, in continual proportionals, each of the terms, except the first and last, is used as the consequent of one ratio, and the antecedent of another; while in other proportionals, each term is used only as an antecedent or consequent, and not as both

9. The second of three continual proportionals is said to be a *mean proportional* between the other two.*

10. When there is any number of magnitudes of the same kind, greater than two, the first is said to have to the last the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratios of A to B, B to C, and C to D.†

11. When three magnitudes are continual proportionals, the ratio of the first to the third is said to be *duplicate* of the ratio of the first to the second, or of the second to the third.

12. When four magnitudes are continual proportionals, the ratio of the first to the fourth is said to be *triplicate* of the ratio of the first to the second, of the ratio of the second to the third, or of the ratio of the third to the fourth.‡

13. In proportionals, the antecedent terms are called *homologous* to one another, as also the consequents to one another.§

"Geometers make use of the following technical words to denote different modes of deriving one proportion from another, by changing either the order or the magnitudes of the terms."

It may be proper to remark, that, by writers on arithmetic and algebra, continual proportionals are often called quantities in *geometrical progression*. This, however, is improper, as they have nothing in their nature to connect them with geometry more than with arithmetic.

The following numbers are continual proportionals, and may serve the learner for illustration:—

3, 6, 12, 24;   54, 18, 6, 2;   4, 6, 9;   64, 48, 36, 27.

* In any series of magnitudes, the first and last are called *extremes*, and the rest *means*. When, however, we speak simply of a mean proportional, it is always to be understood as defined above.

† The meaning of compound ratio,—an expression which has been introduced to prevent circumlocution,—will be better understood by the learner by studying proposition F of this book, the 18th proposition of the Supplement, and the 23d of the sixth book, than by any illustration that could be given in this place.

‡ In continual proportionals, by their own nature, and that of compound ratio, the ratio of the first to the third is compounded of two equal ratios; and the ratio of the first to the fourth, of three equal ratios; and hence we see the reason and the propriety of calling the first duplicate ratio, and the second triplicate. It is plain, that on similar principles, the ratio of the first to the fifth would be said to be *quadruplicate* of the ratio of the first to the second, of the second to the third, &c.; and thus we might form other similar terms at pleasure.

The terms *subduplicate*, *subtriplicate*, and *sesquiplicate*, which are sometimes employed by mathematical writers, are easily understood after the explanations given above. In continual proportionals, the ratio of the first term to the second is said to be *subduplicate* of the ratio of the first to the third, and *subtriplicate* of that of the first to the fourth. Again, if there be four continual proportionals, the ratio of the first to the fourth is said to be *sesquiplicate* of the ratio of the first to the third: or, which amounts to the same, the ratio which is compounded of another ratio and its subduplicate, is sesquiplicate of that ratio.

§ Thus, if A : B :: C : D :: E : F; A, C, E are homologous to one another; and B, D, F to one another.

14. *Alternately:** this word is used when there are four proportionals of the same kind: and it is inferred that the first has the same ratio to the third, which the second has to the fourth; or that the first is to the third, as the second to the fourth: as is shown in the 16th proposition of this book.

15. *By inversion:* when there are four proportionals, and it is inferred, that the second is to the first, as the fourth to the third. Prop. B. Book V.

16. *By composition:* when there are four proportionals, and it is inferred, that the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth. 18th Prop. Book V.

17. *By division:* when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. 17th Prop. Book V.

18. *By conversion:* when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth.† Prop. E, Book V.

19. *Ex æquo*, or *ex equali* (scil. *distantiâ*), *from equality of distance:* when there is any number of magnitudes more than two, and as many others, which, taken two in the one rank, and two in the other, in *direct* order, have the same ratio; and it is inferred that the first has to the last of the first rank the same ratio which the first of the other rank has to the last.‡ This is demonstrated in the 22d proposition of this book.

20. *Ex æquo, inversely:* when there are three or more magnitudes, and as many others, which taken two and two in a *cross* order, have the same ratio; that is, when the first magnitude is to the second in the first rank, as the last but one is to the last in the second rank; and the second to the third of the first rank, as the last but two is to the last but one of the second rank, and so

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* In most editions of Euclid, instead of the simple term *alternately*, there is usually employed *permutando*, *alternando*, *by permutation*, or *alternately*. The proposition, however, which is expressed in substance in this definition, is generally cited by the simple word *alternately*, to the exclusion of the Latin terms, *permutando* and *alternando*, the use of which the taste of the present time properly rejects. For the same reason, in the next four definitions, the words *invertendo*, *componendo*, *dividendo*, and *convertendo*, are omitted.

† The substance of the five preceding definitions may be exhibited briefly in the following manner, the signs + and — denoting addition and subtraction, as has been explained already at the beginning of the second book:—

	Let A : B :: C : D;	or let 15 : 6 :: 10 : 4 : then,
Alternately,	A : C :: B : D;	and 15 : 10 :: 6 : 4 :
By inversion.	B : A :: D : C;	and 6 : 15 :: 4 : 10:
By composition,	A+B : B :: C+D : D;	and 21 : 6 :: 14 : 4 :
By division,	A—B : B :: C—D : D;	and 9 : 6 :: 6 : 4 :
By conversion,	A : A—B :: C : C—D;	and 15 : 9 :: 10 : 6.

‡ Thus, in the two ranks of magnitudes,

A, B, C, D;
P, Q, R, S,

if A is to B, as P to Q; B to C, as Q to R; and C to D, as R to S, the inference, *ex æquo*, is, that A is to D as P to S. The two ranks, 6, 9, 15, 12, and 10, 15, 25, 20, afford an example in numbers; as do also 21, 49, 56, 35, 14, and 15, 35, 40, 25, 10.

on; and it is inferred, as in the preceding definition, that the first is to the last of the first rank, as the first to the last of the other rank.* This is proved in the 23d proposition of this book.

AXIOMS.

1. Like multiples of the same, or of equal magnitudes, are equal to one another.†

2. Those magnitudes of which the same, or equal magnitudes, are like multiples, are equal to one another.‡

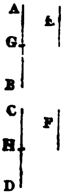
3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.§

PROPOSITIONS.

PROP. I. THEOR.—If any number of magnitudes be like multiples of as many others, each of each; what multiple soever any one of the first is of its part, the same multiple shall all the first magnitudes be of all the others.

Let any number of magnitudes AB, CD be like multiples of as many others E, F, each of each; whatever multiple AB is of E, the same multiple shall AB and CD together be of E and F together.



Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD, equal each of them to F: the number therefore, of the magnitudes CH, HD is equal (hyp.) to the number of the others AG, GB. And because AG is equal to E, and CH to F, therefore AG and CH together are equal (I. ax. 2) to E and F together. For the same reason, GB and HD together are equal to E and F together: wherefore, as many magnitudes as are in AB equal to E, so many are there in AB and CD together equal to E and F together. Therefore, whatever multiple AB is of E, the same multiple are AB and CD together, of E and F together: and the same demonstration would hold, if the number of magnitudes were greater than two. Therefore, &c.

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* Thus, in the two ranks of magnitudes,

A, B, C, D, E;
P, Q, R, S, T,

if A is to B as S to T, B to C as R to S, C to D as Q to R, and D to E as P to Q; then it is inferred, *ex æquo inversely*, that A is to E as P to T. The two ranks, 9, 12, 8, 4, and 18, 9, 6, 8, and the two, 24, 36, 18, 16, 60, and 12, 45, 40, 20, 30, afford examples in numbers.

The learner will perceive, that, in both this definition and the foregoing, the ratio of the first to the last magnitude in the first rank, and the ratio of the first to the last in the second rank, are compounded of equal ratios.

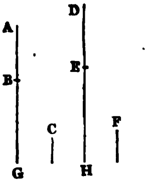
† Or, if equals be multiplied by the same, the products are equal. This axiom might be derived from the second axiom of the first book.

‡ Or, if equals be divided by the same, the quotients are equal.

§ If the teacher and student wish to save time, the Supplement to this book may be used instead of what remains of the book itself.

PROP. II. THEOR.—If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB the first be the same multiple of C the second, that DE the third is of F the fourth; and BG the fifth, the same multiple of C the second, that EH the sixth is of F the fourth: then is AG, the first together with the fifth, the same multiple of C the second, that DH, the third together with the sixth, is of F the fourth.

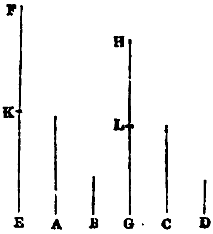


Because AB is the same multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F. In like manner, as many as there are in BG equal to C, so many are there in EH equal to F. As many, then, as are in the whole AG equal to C, so many are there in the whole DH equal to F: therefore, AG is the same multiple of C, that DH is of F; that is, AG, the first and fifth together, is the same multiple of the second C, that DH, the third and sixth together, is of the fourth F. If, therefore, &c.

Cor. From this it is plain, that, if any number of magnitudes be multiples of another C; and as many others be the same multiples of F, each of each; the whole of the first magnitudes is the same multiple of C, that the whole of the last is of F.

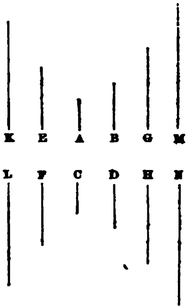
PROP. III. THEOR.—If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken like multiples, these will be like multiples of the second and fourth, each of each.

Let A the first be the same multiple of B the second, that C the third is of D the fourth; and of A, C let the same multiples EF, GH be taken: then EF is the same multiple of B, that GH is of D.



Because EF is the same multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as are in GH equal to C; let EF be divided into the parts EK, KF, each equal to A, and GH into GL, LH, each equal to C: the number therefore of the magnitudes EK, KF, is equal to the number of the others GL, LH. And because A is the same multiple of B, that C is of D, and that EK is equal to A, and GL to C: therefore EK is the same multiple of B, that GL is of D. For the same reason, KF is the same multiple of B that LH is of D; and so, if there be more parts in EF, GH, equal to A, C. Because, therefore, the first EK is the same multiple of the second B, which the third GL is of the fourth D, and that the fifth KF is the same multiple of the second B, which the sixth LH is of the fourth D; EF, the first together with the fifth, is the same multiple (V. 2) of the second B, which GH, the third together with the sixth, is of the fourth D. If, therefore, &c.

PROP. IV. THEOR.—**If the first of four magnitudes have the same ratio to the second which the third has to the fourth; then any like multiples whatever of the first and third have the same ratio to any like multiples of the second and fourth.**



Let A the first have to B the second, the same ratio which the third C has to the fourth D; and of A and C let there be taken any like multiples whatever E, F; and of B and D any like multiples whatever G, H: then E has the same ratio to G, which F has to H.

Take of E and F any like multiples whatever K, L, and of G, H, any like multiples whatever M, N. Then, because E is the same multiple of A, that F is of C; and of E and F have been taken like multiples K, L; therefore (V. 3) K is the same multiple of A, that L is of C. For the same reason, M is the same multiple of B, that N is of D. And because (hyp.) as A is to B, so is C to D, and of A and C have been taken certain like multiples K, L; and of B and D have been taken certain like multiples M, N; if therefore (V. def. 5) K be greater than M, L is greater than N: and if equal, equal; if less, less. But K, L are any like multiples whatever of E, F; and M, N any whatever of G, H: therefore (V. def. 5) as E : G :: F : H. Therefore, &c.

Cor. Likewise, if the first have the same ratio to the second, which the third has to the fourth, then also any like multiples whatever of the first and third have the same ratio to the second and fourth: and in like manner, the first and the third have the same ratio to any like multiples whatever of the second and fourth.

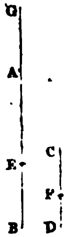
Let A the first have to B the second, the same ratio which the third C has to the fourth D, and of A and C let E and F be any like multiples whatever; then E : B :: F : D.

Take of E, F any like multiples whatever K, L, and of B, D any like multiples whatever G, H. Then it may be demonstrated, as before, that K is the same multiple of A, as L is of C: and, because A : B :: C : D, and of A and C like multiples K and L have been taken; and of B and D like multiples G and H; therefore (V. def. 5) if K be greater than G, L is greater than H; if equal, equal; and if less, less. And K, L are any like multiples of E, F; and G, H any whatever of B, D: as therefore (V. def. 5) E : B :: F : D; and in the same way the other case is demonstrated.

Prop. V. Theor.—If one magnitude be the same multiple of another, which a part taken from the first is of a part taken from the other; the first remainder is the same multiple of the second, that the first magnitude is of the second.

Let the magnitude AB be the same multiple of CD, that AE, taken from the first, is of CF, taken from the other; the remainder EB is the same multiple of the remainder FD, that the whole AB is of the whole CD.

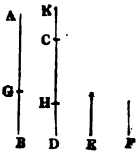
Take AG the same multiple of FD, that AE is of CF: therefore (V. 1) AE is the same multiple of CF, that GE is of CD. But (hyp.) AE is the same multiple of CF, that AB is of CD: therefore EG is the same multiple of CD that AB is of CD; wherefore (V. ax. 1) EG is equal to AB. Take from them the common magnitude AE; and the remainder AG is equal to the remainder EB. Wherefore, since AE is the same multiple of CF, that AG is of FD, and that AG is equal to EB; therefore AE is the same multiple of CF, that EB is of FD. But AE is the same multiple of CF, that AB is of CD: therefore EB is the same multiple of FD, that AB is of CD. Therefore, if one magnitude, &c.



Prop. VI. Theor.—If two magnitudes be like multiples of two others, and if like multiples of these be taken from the first two, the remainders are either equal to these others, or are like multiples of them.

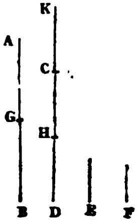
Let the two magnitudes AB, CD be like multiples of the two E, F, and AG, CH taken from the first two, be like multiples of the same E, F; the remainders GB, HD are either equal to E, F, or are like multiples of them.

First, let GB be equal to E; HD is equal to F. Make CK equal to F; and because AG is the same multiple of E, that CH is of F, and that GB is equal to E, and CK to F; therefore AB is the same multiple of E, that KH is of F. But (hyp.) AB is the same multiple of E that CD is of F: therefore KH is the same multiple of F, that CD is of F; wherefore (V. ax. 1) KH is equal to CD.



Take away CH, then the remainder KC is equal to the remainder HD: but KC is equal to F; HD therefore is equal to F.

But let GB be a multiple of E; then HD is the same multiple of F. Make CK the same multiple of F, that GB is of E. Then because AG is the same multiple of E, that CH is of F; and GB the same multiple of E, that CK is of F: therefore (V. 2) AB is the same multiple of E, that KH is of F. But AB is the same multiple of E, that CD is of F: therefore KH is the same multiple of F, that CD is of it: wherefore (V. ax. 1) KH is equal to CD. Take away CH from both: therefore the remainder KC is equal to the remainder HD: and because GB is the same multiple of E, that KC is of F, and that KC is equal to HD; therefore HD is the same multiple of F, that GB is of E. If, therefore, two magnitudes, &c.



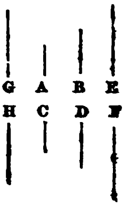
PROP. A. THEOR.—If four magnitudes be proportional; then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Take any like multiples of each of them, as the doubles of each; then (V. def. 5) if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth; but if the first be greater than the second, the double of the first is greater (V. ax. 3) than the double of the second; wherefore also the double of the third is greater (V. def. 5) than the double of the fourth: therefore the third is greater (V. ax. 4) than the fourth. In like manner, if the first be equal to the second, or less than it, the third can be proved to be equal to the fourth or less than it. Therefore, if four magnitudes, &c.

PROP. B. THEOR.—If four magnitudes be proportionals, they are proportionals also, when taken inversely.

If A : B :: C : D; then also inversely B : A :: D : C.

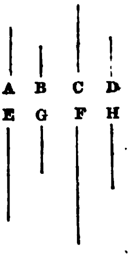
Take of B and D any like multiples E and F; and of A and C any like multiples G and H. First, let E be greater than G; and, because A : B :: C : D, and of A and C, the first and third, G and H are like multiples; and of B and D, the second and fourth, E and F are like multiples; and that G is less than E, H is also (V. def. 5) less than F; that is, F is greater than H: if, therefore, E be greater than G, F is greater than H. In like manner, if E be equal to G, F may be shown to be equal to H; and, if less, less; and E, F are any like multiples whatever of B and D, and G, H, any whatever of A and C: therefore, as B : A :: D : C. If, then, &c.



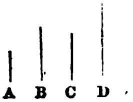
PROP. C. THEOR.—If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second as the third is to the fourth.

Let the first A be the same multiple of B the second, that C the third is of D the fourth; then A : B :: C : D.

Take of A and C any like multiples whatever E and F; and of B and D any like multiples whatever G and H. Then, because A is the same multiple of B that C is of D; and that E is the same multiple of A, as F is of C; E is the same multiple (V. 3) of B, that F is of D: therefore E and F are the same multiples of B and D. But G and H are like multiples of B and D; therefore if E be a greater multiple of B, than G is, F is a greater multiple of D than H is; that is, if E be greater than G, F is greater than H. In like manner, if E be equal to G, or less, F is equal to H, or less than it. But E, F are like multiples, any whatever, of A, C; and G, H any like multiples whatever of B, D. Therefore (V. def. 5) A : B :: C : D.



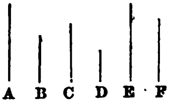
Next, let the first A be the same part of the second B, that the third C is of the fourth D; A : B :: C : D. For B is the same multiple of A, that D is of C: wherefore, by the preceding case, B : A :: D : C; and inversely (V. B) A : B :: C : D. If, therefore, &c.



PROP. D. THEOR.—If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or part of the fourth.

Let A : B :: C : D; and first let A be a multiple of B; C is the same multiple of D.

Take E equal to A, and whatever multiple A or E is of B, make F the same multiple of D. Then because A : B :: C : D; and of B the second, and D the fourth like multiples E and F have been taken; then (V. 4, cor.) A : E :: C : F. But A is equal to E; therefore C (V. A) is equal to F; and F is the same multiple of D that A is of B: wherefore C is the same multiple of D, that A is of B.



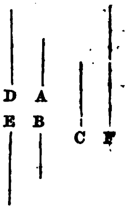
Next, let the first A (see last figure for prop. C) be a part of the second B; C the third is the same part of the fourth D.

Because A : B :: C : D; then, inversely, (V. B) B : A :: D : C. But A is a part of B, therefore B is a multiple of A; and, by the preceding case, D is the same multiple of C, that is, C is the same part of D, that A is a part of B. If, therefore, &c.

PROP. VII. THEOR.—Equal magnitudes have the same ratio to the same magnitude of the same kind: and the same has the same ratio to equal magnitudes of the same kind.

Let A and B be equal magnitudes, and C any other of the same kind: A and B have each the same ratio to C, and C has the same ratio to each of the magnitudes A and B.

Take of A and B any like multiples whatever D and E, and of C any multiple whatever F. Then, because D is the same multiple of A, that E is of B, and that A is equal to B; D is equal (V. ax. 1) to E: therefore, if D be greater than F, E is greater than F; if equal, equal; and if less, less. But D, E are any like multiples of A, B; and F is any multiple of C. Therefore (V. def. 5) as A : C :: B : C.

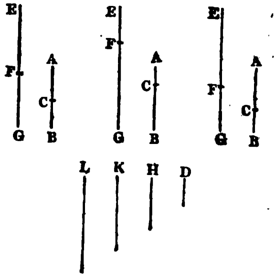


Likewise C has the same ratio to A, that it has to B. For, the same construction being made, D may in like manner be shown to be equal to E: therefore, if F be greater than D, it is likewise greater than E; if equal, equal; and if less, less. But F is any multiple whatever of C; and D, E are any like multiples whatever of A, B: therefore (V. def. 5) C : A :: C : B. Therefore, equal magnitudes, &c.

PROP. VIII. THEOR.—Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than to the greater.

Let AB, BC be unequal magnitudes, of which AB is the greater, and let D be any magnitude whatever: AB has a greater ratio to D than BC to D: and D has a greater ratio to BC than to AB.

If the magnitude which is not the greater of the two AC, CB, be not less than D, take EF, FG, the doubles of AC, CB, as in Fig. 1. But, if that which is not the greater of the two AC, CB, be less than D (as in Figs. 2 and 3), this magnitude can be multiplied so as to become greater than D, whether it be AC, or CB. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the same multiple of CB. Therefore EF and FG are each of them greater than D: and in each of the cases take H the double of D, K its triple, and so on, till the multiple of D be that which first becomes greater than FG. Let L be that multiple of D which is first greater than FG, and K the multiple of D which is next less than L.

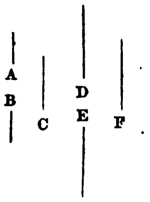


Then, because L is the multiple of D, which is the first that becomes greater than FG, the next preceding multiple K is not greater than FG; that is, FG is not less than K: and since EF is the same multiple of AC, that FG is of CB: FG is the same multiple (V. 1) of CB, that EG is of AB: wherefore EG and FG are like multiples of AB and CB; and it was shown, that FG was not less than K, and, by the construction, EF is greater than D; therefore the whole EG is greater than K and D together. But K together with D is equal to L: therefore EG is greater than L; but FG is not greater than L; and EG, FG are like multiples of AB, BC, and L is a multiple of D; therefore (V. def. 7) AB has to D a greater ratio than BC has to D.

Also D has to BC a greater ratio than it has to AB. For, the same construction being made, it may be shown, in like manner, that L is greater than FG, but that it is not greater than EG: and L is a multiple of D; and FG, EG are like multiples of CB, AB: therefore (V. def. 7) D has a greater ratio to CB than to AB: wherefore, of unequal magnitudes, &c.

PROP. IX. THEOR.—Magnitudes which have the same ratio to the same magnitude are equal: and those to which the same magnitude has the same ratio are equal.

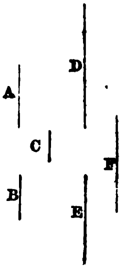
Let A, B have each of them the same ratio to C: A is equal to B. For, if they be not equal, one of them is greater than the other; let A be the greater. Then, by what was shown in the preceding proposition, there are some like multiples of A and B, and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let such multiples be taken, and let D, E be the multiples of A, B, and F the multiple of C, so that D may be greater than F, and E not greater than F. But because A is to C as B is to C, and of A, B like multiples D, E are taken, and of C a multiple F is taken; and that D is greater than F; E shall also (V. def. 5) be greater than F; but E is not greater than F, which is impossible; A, therefore, and B are not unequal, that is, they are equal.



Next, let C have the same ratio to each of the magnitudes A and B; A and B are equal. For, if they be not, one of them is greater than the other; let A be the greater. Therefore, as was shown in proposition 8th, there is some multiple F of C, and some like multiples E and D, of B and A, such that F is greater than E, and not greater than D; but because C : B :: C : A, and that F, the multiple of the first, is greater than E, the multiple of the second; F the multiple of the third, is greater (V. def. 5) than D, the multiple of the fourth: but F is not greater than D, which is impossible. Therefore A is equal to B: wherefore, magnitudes which, &c.

PROP. X. THEOR.—That magnitude which has a greater ratio than another has to the same magnitude, is the greater of the two: and that magnitude, to which the same has a greater ratio than it has to another magnitude, is the less of the two.

Let A have to C a greater ratio than B has to C; A is greater than B. For, since A has a greater ratio to C, than B has to C, there are (V. def. 7) some like multiples of A and B, and some multiple of C, such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it: let them be taken, and let D, E be like multiples of A, B, and F a multiple of C, such, that D is greater than F, but E is not greater than F. Therefore D is greater than E: and because D and E are like multiples of A and B, and D is greater than E: therefore (V. ax. 4) A is greater than B.

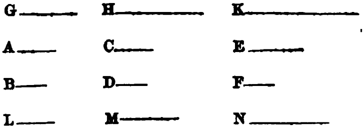


Next, let C have a greater ratio to B than it has to A; B is less than A. For (V. def. 7) there is some multiple F of C, and some like multiples E and D of B and A, such, that F is greater than E, but is not greater than D: E therefore is less than D; and because E and D are like multiples of B and A, therefore (V. ax. 4) B is less than A. That magnitude, therefore, &c.

Prop. XI. Theor.—Ratios that are equal to the same ratio, are equal to one another.

Let A : B :: C : D, and C : D :: E : F; then A : B :: E : F.

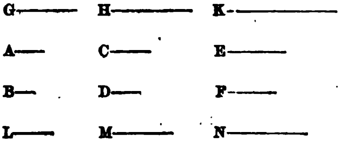
Take of A, C, E, any like multiples whatever, G, H, K; and of B, D, F, any like multiples whatever L, M, N. Therefore, (V. def. 5) since A : B :: C : D, and G, H are taken like multiples of A, C, and L, M of B, D; if G be greater than L, H is greater than M; if equal, equal; and if less, less. Again, because C : D :: E : F, and H, K are taken like multiples of C, E; and M, N, of D, F; if H be greater than M, K is greater than N; if equal, equal; and if less, less. But, if G be greater than L, it has been shown, that H is greater than M; if equal, equal; and if less, less: therefore if G be greater than L, K is greater than N; if equal, equal; and if less, less: and G, K are any like multiples whatever of A, E; and L, N any whatever of B, F. Therefore (V. def. 5) A : B :: E : F: wherefore, ratios, &c.



Prop. XII. Theor.—If any number of magnitudes be proportionals, as any one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents.

Let any number of magnitudes A, B, C, D, E, F be proportionals; that is, as A : B :: C : D :: E : F; then as A is to B, so are A, C, E together to B, D, F together.

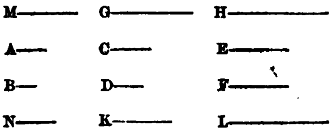
Take of A, C, E any like multiples whatever G, H, K: and of B, D, F any like multiples whatever L, M, N: then (V. def. 5) because A is to B, as C is to D, and as E to F, and that G, H, K are like multiples of A, C, E; and L, M, N like multiples of B, D, F; if G be greater than L, H is greater than M, and K greater than N; if equal, equal; and if less, less. Therefore, if G be greater than L, then G, H, K together are greater than L, M, N together; if equal, equal; and if less, less. And G, and G, H, K together are any like multiples of A, and A, C, E together; because if there be any number of magnitudes like multiples of as many, each of each, whatever multiple one of them is of its part, the same multiple (V. 1) is the whole of the whole. For the same reason L, and L, M, N are any like multiples of B, and B, D, F. As therefore (V. def. 5) A is to B, so are A, C, E together, to B, D, F together: wherefore, if any number, &c.



PROP. XIII. THEOR.—If the first have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first will also have to the second a greater ratio than the fifth has to the sixth.

Let A the first, have the same ratio to B the second, which C the third, has to D the fourth, but C the third, to D the fourth, a greater ratio than E the fifth, to F the sixth: A has also to B a greater ratio than E to F.

Because C has a greater ratio to D, than E to F, there are (V. def. 7) some like multiples of C and E, and some of D and F, such, that the multiple of C is greater than that of D, but the multiple of E is not greater than that of F. Let such be taken, and of C, E let G, H be like multiples, and K, L like multiples of D, F, so that G be greater than K, but H not greater than L; and whatever multiple G is of C, take M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B. Then (V. def. 5) because A : B :: C : D, and of A and C, M and G are like multiples; and of B and D, N and K are like multiples; if M be greater than N, G is greater than K; if equal, equal; and if less, less; but G is greater than K, therefore M is greater than N. But H is not greater



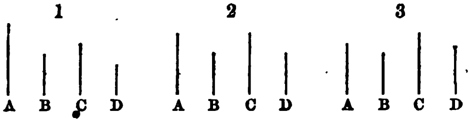
than L; and M, H are like multiples of A, E; and N, L like multiples of B, F: therefore (V. def. 7) A has a greater ratio to B, than E has to F. Wherefore, if the first, &c.

Cor. If the first have a greater ratio to the second, than the third has to the fourth, but the third the same ratio to the fourth, which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second, than the fifth has to the sixth.

PROP. XIV. THEOR.—If the first have to the second the same ratio which the third has to the fourth; then if the first be greater than the third, the second is greater than the fourth; if equal, equal; and if less, less.

Let A : B :: C : D; then if A be greater than C, B is greater than D.

Because A is greater than C, and B is any other magnitude, A has (V. 8) to B a greater ratio than C to B: but A : B :: C : D;



therefore also (V. 13) C has to D a greater ratio than C has to B. But (V. 10) of two magnitudes, that to which the same has the greater ratio is the less: wherefore D is less than B.

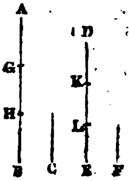
Secondly, if A be equal to C, B is equal to D: for A is to B, as C, that is A, to D; B therefore (V. 9) is equal to D.

Thirdly, if A be less than C, B shall be less than D. For C is greater than A; and because C : D :: A : B, D is greater than B, by the first case. Therefore, if the first, &c.

PROP. XV. THEOR.—Magnitudes have the same ratio to one another that their like multiples have.

Let AB be the same multiple of C, that DE is of F: C is to F, as AB to DE.

Because AB is the same multiple of C, that DE is of F; there are as many magnitudes in AB each equal to C, as there are in DE each equal to F: let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE. Then the number of the first AG, GH, HB, shall be equal to the number of the last DK, KL, LE. And because AG, GH, HB are all equal, and that DK, KL, LE are also equal to one another; therefore (V. 7) AG : DK :: GH : KL, and as HB : LE. Also (V. 12), as one of the antecedents to its consequent, so are all the antecedents together to all

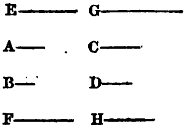


the consequents together; wherefore as AG : DK :: AB : DE; but AG is equal to C, and DK to F. Therefore, as C : F :: AB : DE; wherefore magnitudes, &c.

PROP. XVI. THEOR.—If four magnitudes of the same kind be proportionals, they are also proportionals when taken alternately.

Let A : B :: C : D; then, alternately, A : C :: B : D.

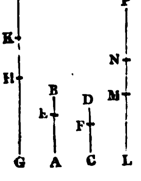
Take of A and B any like multiples whatever E and F; and of C and D take any whatever G and H: and because E is the same multiple of A, that F is of B, and that magnitudes (V. 15) have the same ratio to one another which their like multiples have; therefore A : B :: E : F; but as A : B :: C : D; wherefore (V. 11) as C : D :: E : F. Again (V. 15) because G, H are like multiples of C, D, as C : D :: G : H; but as C : D :: E : F; wherefore (V. 11) as E : F :: G : H. But (V. 14) when four magnitudes are proportionals, if the first be greater than the third, the second is greater than the fourth; if equal, equal; and if less, less: therefore, if E be greater than G, F likewise is greater than H; if equal, equal; and if less, less: and E, F are any like multiples whatever of A, B; and G, H any whatever of C, D. Therefore (V. def. 5) A : C :: B : D; wherefore, if four magnitudes, &c.



PROP. XVII. THEOR.—If four magnitudes be proportionals; then, by division, the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth.

Let AB : BE :: CD : DF; then AE : EB :: CF : FD.

Take of AE, EB, CF, FD any like multiples whatever GH, HK, LM, MN; and, again, of EB, FD, take any like multiples whatever KX, NP. Then, because GH is the same multiple of AE, that HK is of EB; therefore (V. 1) GH is the same multiple of AE, that GK is of AB; but GH is the same multiple of AE, that LM is of CF; wherefore GK is the same multiple of AB, that LM is of CF. Again, because LM is the same multiple of CF, that MN is of FD; therefore (V. 1) LM is the same multiple of CF, that LN is of CD. But LM was shown to be the same multiple of CF, that GK is of AB; GK therefore is the same multiple of AB, that LN is of CD; that is, GK, LN are like multiples of AB, CD. Next, because HK is the same multiple of EB, that MN is of FD; and that KX is also the same multiple of EB, that NP is of FD; therefore (V. 2) HX is the same multiple of EB, that MP is of FD. And, because AB : BE :: CD : DF, and that of AB and CD, GK and LN are like multiples, and of



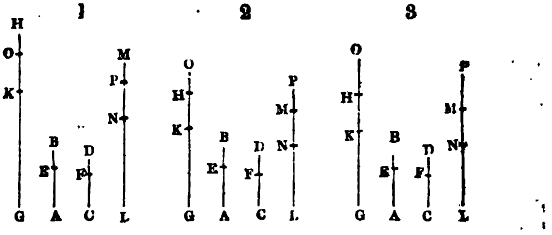
EB and FD, HX and MP are like multiples; if GK be greater than HX, then (V. def. 5) LN is greater than MP; if equal, equal; and if less, less. But if GH be greater than KX, by adding the common part HK to both, GK is greater than HX; wherefore also LN is greater than MP; and by taking away MN from both, LM is greater than NP: therefore, if GH be greater than KX, LM is greater than NP. In like manner it may be demonstrated, that if GH be equal to KX, LM is likewise equal to NP; and if less, less: and GH, LM are any like multiples whatever of AE, CF, and KX, NP are any whatever of EB, FD. Therefore (V. def. 5) as AE : EB :: CF : FD. Wherefore, if four magnitudes, &c.

PROP. XVIII. THEOR.—If four magnitudes be proportionals; then, by composition, the first and second together are to the second, as the third and fourth together are to the fourth.

Let AE : EB :: CF : FD; then, AB : BE :: CD : DF.

Take of AB, BE, CD, DF any like multiples GH, HK, LM, MN; and again, of BE, DF, take any like multiples KO, NP. Then, because KO, NP are like multiples of BE, DF; and that KH, NM are also like multiples of BE, DF, if KO, the multiple of BE, be greater than KH, which is a multiple of the same BE, NP, likewise the multiple of DF, shall be greater than MN, the multiple of the same DF; and if KO be equal to KH, NP shall be equal to NM; and if less, less.

First, Let KO not be greater than KH, therefore NP is not greater than NM: and because GH, HK are like multiples of AB, BE, and that AB is greater than BE, therefore (V. ax. 3) GH is greater than HK; but KO is not greater than KH, wherefore GH is greater than KO. In like manner, it may be shown, that LM is greater than NP. Therefore, if KO be not greater than KH, then GH, the multiple of AB, is always greater than KO, the multiple of BE; and likewise LM, the multiple of CD, greater than NP, the multiple of DF.



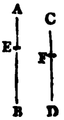
Next, let KO be greater than KH; therefore, as has been shown, NP is greater than NM. And because the whole GH is the same multiple of the whole AB, that HK is of BE, the remainder GK is the same multiple (V. 5) of the remainder AE, that GH is of AB: which is the same that LM is of CD. In like manner, because LM is the same multiple of CD, that MN is of DF, the remainder LN is the same multiple (V. 5) of the remainder CF, that the whole LM is of the whole CD. But it was shown that LM is the same multiple of CD, that GK is of AE; therefore GK is the same multiple of AE, that LN is of CF; that is, GK, LN are like multiples of AE, CF. And because KO, NP are like multiples of BE, DF, if from KO, NP there be taken KH, NM, which are also like multiples of BE, DF, the remainders HO, MP are (V. 6) either equal to BE, DF, or like multiples of them. First, let HO, MP be equal to BE, DF; and because AE : EB :: CF : FD, and that GK, LN are like multiples of AE, CF; then (V. 4, cor.) GK : EB :: LN : FD; but HO is equal toEB, and MP to FD; wherefore GK : HO :: LN : MP. Therefore (V. E) if GK be greater than HO, LN is greater than MP; if equal, equal; and if less, less.

But let HO, MP be like multiples of EB, FD: and because AE : EB :: CF : FD, and that of AE, CF are taken like multiples GK, LN; and of EB, FD, like multiples HO, MP; therefore (V. def. 5) if GK be greater than HO, LN is greater than MP; if equal, equal; and if less, less; which was likewise shown in the preceding case. If therefore GH be greater than KO; taking KH from both, GK is greater than HO; wherefore also LN is greater than MP; and consequently adding NM to both, LM is greater than NP: therefore, if GH be greater than KO, LM is greater than NP. In like manner it may be shown that if GH be equal to KO, LM is equal to NP; and if less, less. And in the case in which KO is not greater than KH, it has been shown that GH is always greater than KO, and likewise LM than NP: but GH, LM are any like multiples of AB, CD, and KO, NP are any whatever of BE, DF; therefore (V. def. 5) as AB : BE :: CD : DF. If, therefore, four magnitudes, &c.

Prop. XIX. Theor.—If the first be to the second, as a part of the first to a part of the second; the remaining part of the first is to the remaining part of the second, as the first to the second.

Let AB : CD :: AE : CF; then EB : FD :: AB : CD.

Because AB : CD :: AE : CF; alternately (V. 16) BA : AE :: DC : CF: and therefore (V. 17) by division, as BE : EA :: DF : FC; and alternately, as BE : DF :: EA : FC. But (hyp.) as AE : CF :: AB : CD; therefore (V. 11) also BE : DF :: AB : CD; wherefore, if the first, &c.



Cor. If the first be to the second, as a part of the first is to a part of the second: the remaining part of the first is to the remaining part of the second, as the other part of the first to the remaining part of the second. The demonstration is contained in the preceding.

Prop. E. Theor.—If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB : BE :: CD : DF; then BA : AE :: DC : CF. (See the figure to the last proposition.)

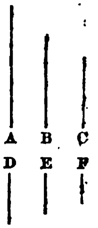
Because AB : BE :: CD : DF, by division (V. 17) AE : EB :: CF : FD; and (V. B) by inversion, BE : EA :: DF : FC. Wherefore (V. 18) by composition, BA : AE :: DC : CF. If, therefore, four magnitudes, &c.

Cor. It would be proved in a similar manner, that the first is to the sum of the first and second, as the third to the sum of the third and fourth.

Prop. XX. Theor.—If there be three magnitudes, and other three, which, taken two and two, have the same ratio; if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the same ratio, viz. A : B :: D : E; and B : C :: E : F; if A be greater than C, D is greater than F; if equal, equal; and if less, less.

Because A is greater than C, and B is any other magnitude, and that (V. 8) the greater has to the same magnitude a greater ratio than the less has; therefore A has to B a greater ratio than C has to B. But as D : E :: A : B; therefore (V. 13) D has to E a greater ratio than C to B; and because B : C :: E : F, by inversion, C : B :: F : E; and D was shown to have to E a greater ratio than C to B; therefore (V. 13, cor.) D has to E a greater ratio than F to E; but (V. 10) the magnitude which has a greater ratio than another to the same magnitude, is the greater of the two: D is therefore greater than F.



Secondly, let A be equal to C; D is also equal to F. Because A and C are equal to one another (V. 7), A : B :: C : B; but A : B :: D : E; and C : B :: F : E; wherefore also (V. 11) D : E :: F : E; and therefore (V. 9) D is equal to F.

Next, let A be less than C; D is also less than F. For C is greater than A, and, as was shown in the first case, C : B :: F : E, and in like manner B : A :: E : D; therefore F is greater than D, by the first case; that is, D is less than F. Therefore, if there be three magnitudes, &c.

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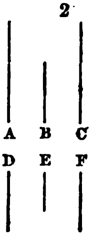

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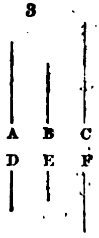
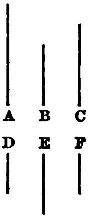


PROP. XXI. THEOR.—If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, which have the same ratio, taken two and two, but in a cross order, viz. as A : B :: E : F, and as B : C :: D : E. If A be greater than C, D is greater than F; if equal, equal; if less, less.

Because A is greater than C, and B is any other magnitude, A has to B a greater ratio (V. 8) than C has to B; but as E : F :: A : B; therefore (V. 13) E has to F a greater ratio than C to B. And because B : C :: D : E, by inversion, C : B :: E : D; and E was shown to have to F a greater ratio than C to B; therefore (V. 13, cor.) E has to F a greater ratio than E to D; but the magnitude to which the same has a greater ratio than it has to another, is (V. 10) the less of the two: F therefore is less than D; that is, D is greater than F.



Secondly, let A be equal to C; D shall be equal to F. Because A and C are equal (V. 7) A : B :: C : B; but A : B :: E : F; and C : B :: E : D; wherefore (V. 11) E : F :: E : D; and therefore (V. 9) D is equal to F.

Next, let A be less than C; D is also less than F. For C is greater than A, and, as was shown, C : B :: E : D, and in like manner, B : A :: F : E; therefore F is greater than D, by case first; that is, D is less than F. Therefore, if there be three magnitudes, &c.

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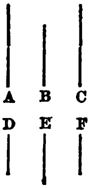

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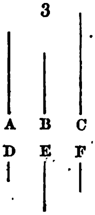
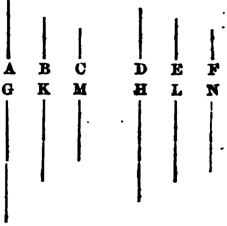


PROP. XXII. THEOR.—If there be any number of magnitudes more than two, and as many others, which, taken two and two in order, have the same ratio; the first has to the last of the first magnitudes the same ratio which the first has to the last of the others. N. B. This is usually cited by the words "*ex æquo*," or "*ex æquali*."

First, let there be three magnitudes A, B, C, and as many others D, E, F, which, taken two and two, have the same ratio, that is, such that A : B :: D : E; and B : C :: E : F; then A : C :: D : F.

Take of A, D, any like multiples whatever, G, H; of B, E, any whatever K, L; and of C, F any whatever M, N. Then, because A : B :: D : E, and that G, H are like multiples of A, D, and K, L of B, E; (V. 4) as G : K :: H : L. For the same reason, K : M :: L : N; and (V. 20) because there are three magnitudes G, K, M, and other three H, L, N, which, two and two, have the same ratio; if G be greater than M, H is greater than N; if equal, equal; and if less, less; and G, H are any like multiples whatever of A, D, and M, N of C, F; therefore (V. def. 5) as A : C :: D : F.



Next, let there be four magnitudes, A, B, C, D, and other four, E, F, G, H, which two and two have the same ratio, viz. as A : B :: E : F; as B : C :: F : G; and as C : D :: G : H: then A : D :: E : H.

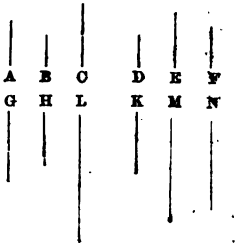
A.B.C.D.
E.F.G.H.

Because A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio; by the foregoing case, A : C :: E : G. But C : D :: G : H; wherefore again, by the first case, A : D :: E : H; and so on, whatever is the number of magnitudes. Therefore, if there be any number of magnitudes, &c.

PROP. XXIII. THEOR.—If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first has to the last of the first magnitudes the same ratio which the first has to the last of the others. N.B. This is usually cited by the words, "*ex æquali in proportione perturbata;*" or, "*ex æquo perturbato;*" or, "*ex æquo inversely.*"

First, let there be three magnitudes A, B, C, and other three D, E, F, which, taken two and two, in a cross order, have the same ratio; that is, such, that A : B :: E : F; and B : C :: D : E; then A : C :: D : F.

Take of A, B, D any like multiples whatever G, H, K; and of C, E, F any like multiples whatever L, M, N. Then, because G, H are like multiples of A, B, and that magnitudes (V. 15) have the same ratio which their like multiples have; as A : B :: G : H. For the same reason, as E : F :: M : N. But as A : B :: E : F; as therefore (V. 11) G : H :: M : N. And because B : C :: D : E, and that H, K are like multiples of B, D, and L, M of C, E; then (V. 4) as H : L :: K : M: and it has been shown that G : H :: M : N. Then, (V. 21) because there are



three magnitudes G, H, L, and other three K, M, N, which have the same ratio taken two and two in a cross order; if G be greater than L, K is greater than N; if equal, equal; and if less, less; and G, K are any like multiples whatever of A, D; and L, N of C, F; as, therefore, (V. def. 5) A : C :: D : F.

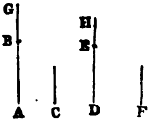
Next, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which taken two and two in a cross order, have the same ratio, viz. A : B :: G : H; B : C :: F : G: and C : D :: E : F: then A : D :: E : H.

A.B.C.D.
E.F.G.H.

Because A, B, C, are three magnitudes, and F, G, H other three, which, taken two and two in a cross order, have the same ratio; by the first case, A : C :: F : H. But C : D :: E : F; wherefore again, by the first case, A : D :: E : H: and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c.

PROP. XXIV. THEOR.—If the first have to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together have to the second, the same ratio which the third and sixth together have to the fourth.

Let AB : C :: DE : F; and let BG the fifth have to C the second, the same ratio which EH the sixth has to F the fourth; then, AG, the sum of the first and fifth, has to C the second, the same ratio which DH, the sum of the third and sixth, has to F the fourth.



Because BG : C :: EH : F; by inversion, C : BG :: F : EH: and because AB : C :: DE : F, and C : BG :: F : EH; ex æquo (V. 22), AB : BG :: DE : EH; and, by composition, (V. 18) as AG : GB :: DH : HE; but as GB : C :: HE : F. Therefore, ex æquo, as AG : C :: DH : F: wherefore, if the first, &c.

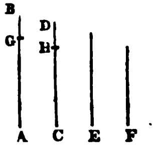
Cor. 1. If the same hypothesis be made as in the proposition, the difference of the first and fifth shall be to the second, as the difference of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition manifestly holds true, of two ranks of magnitudes, whatever is their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude.

PROP. XXV. THEOR.—If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together.

Let AB : CD :: E : F; and let AB be the greatest, and consequently (V. 14, and A) F the least. AB and F are together greater than CD, together with E.

Take AG equal to E, and CH equal to F. Then because AB : CD :: E : F, and that AG is equal to E, and CH equal to F; AB : CD :: AG : CH. And (V. 19) because the whole AB is to the whole CD, as AG is to CH, likewise the remainder GB is to the remainder HD, as the whole AB is to the whole CD. But AB is greater than CD, therefore (V. A.) GB is greater than HD; and because AG is equal to E, and CH to F; AG and F together are equal to CH and E together. If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD; AB and F together are greater than CD and E. Therefore, if four magnitudes, &c.



Cor. The sum of the extremes of three continual proportionals is greater than double the mean.

Prop. F. Theor.—Ratios which are compounded of the same ratios, are equal to one another.

Let A : B :: D : E; and B : C :: E : F; the ratio which is compounded of the ratios of A to B, and B to C, which, by the definition of compound ratio, is the ratio of A to C, is the same with the ratio of D to F, which, by the same definition, is compounded of the ratios of D to E, and E to F.

A.B.C.
D.E.F.

Because there are three magnitudes A, B, C, and three others D, E, F, which, taken two and two in order, have the same ratio; ex æquo, (V. 22) A : C :: D : F.

Next, let A : B :: E : F, and B : C :: D : E; therefore, *ex æquo inversely*, (V. 23) A : C :: D : F: and in like manner the proposition may be demonstrated, whatever be the number of ratios. Therefore ratios, &c.

A.B.C.
D.E.F.

BOOK V.

SUPPLEMENT.

[THE theory of proportion has now been delivered in the accurate but prolix method given by Euclid. It may suit the views and convenience, however, of many teachers and learners, to have its principal properties established in a more concise and condensed form, as in the following Supplement, so as to save time which may be more profitably devoted to other researches. For this purpose, the notation and some of the simplest principles of algebra may be advantageously employed; and should the student be unacquainted with that science, it will be necessary for him to commence by studying the following explanations and principles.]

DEFINITIONS AND PRINCIPLES.*

1. The sign +, read *plus*, † denotes addition. Thus, the expression A+B signifies the sum of the quantities represented by A and B; so that if A represent 5 feet, and B 3 feet, A+B denotes 8 feet.

2. The sign —, read *minus*, signifies, that the quantity after it is taken from the one before it. Thus A—B expresses what remains when B is taken from A.

3. The product arising from multiplying one number by another is expressed by writing the letters representing them, one after the other, without any sign between them; and sometimes by placing between them a point, or the sign ×. Thus, if A and B represent two numbers, their product is denoted by AB, by A.B, or by A×B; and A and B are called the *factors* of this product. The product of this result by the number C, is denoted by ABC, by A.B.C, or by A×B×C: and the product of a greater number of factors is expressed in a similar way. In like manner, 2A denotes twice A, 3A three times A, &c.

4. Hence, conversely, if AB be divided by B, the quotient is A.

5. A product is called a *power*, when the factors are all the same. Thus, AA, or as it is generally written, A^2, is called the

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* In this Supplement, for brevity, these are referred to, as No. 1, No. 2, &c.

† A few of these characters have been explained already in page 43. The explanation is repeated here with the view of presenting in connexion all that is necessary on the subject.

second power, or the square of A; AAA, or A^3, the third power, or cube of A; AAAA, or A^4, its fourth power, &c.

In relation to these powers, A is called their *root*. Thus, A is the second or square root of A^2, the third or cube root of A^3, the fourth root of A^4, &c. In like manner, the *second* or *square root* of A is a number which, when multiplied by itself, produces A; the *third* or *cube root* of A is such a number, that if it be multiplied by itself, and the product by the same root again, the final product will be A. The square root of A is denoted by $\sqrt{A}$ or $A^{\frac{1}{2}}$, its cube root by $\sqrt[3]{A}$, or $A^{\frac{1}{3}}$, its fourth root by $A^{\frac{1}{4}}$, &c.

6. The quotient arising from dividing one number by another is denoted by writing the dividend as the numerator of a fraction, and the divisor as its denominator. Thus, $\frac{A}{B}$ denotes the quotient obtained by dividing the number represented by A, by the one represented by B, or it is the number which shows how often B is contained in A.

7. The signs $=$, $>$, $<$, signify respectively *equal to*, *greater than*, *less than*. Thus $A=B$ denotes that A is equal to B; $A>B$, that A is greater than B; and $A<B$, that A is less than B: the opening of the angle in the last two being turned towards the greater magnitude.

The principles employed in this Supplement, in addition to the axioms of the first and fifth books, are the following.

8. A fraction is *multiplied* by a number, by multiplying its numerator by that number. Thus, if $\frac{2}{7}$ be multiplied by 3, the product is $\frac{6}{7}$.

9. A fraction is *divided* by a number by multiplying its denominator by that number. Thus, $\frac{7}{3}$ is evidently twice as great as $\frac{7}{6}$; the one being a third of 7, and the other only a sixth of it.

10. The value of a fraction is not changed by multiplying or dividing both the numerator and denominator by the same number. Thus, $\frac{5}{7}$ is the same as $\frac{10}{14}$.

11. Fractions having a common denominator, are *added* or *subtracted* by adding or subtracting their numerators, and retaining the common denominator. Thus, the sum of $\frac{7}{11}$ and $\frac{2}{11}$ is $\frac{9}{11}$, and their difference is $\frac{5}{11}$.

12. To find the *product* of any number of fractions, multiply their numerators together for the numerator of the required fraction, and their denominators for its denominator. Thus, the product of $\frac{2}{3}$ and $\frac{4}{5}$ is $\frac{8}{15}$. In this operation, since $\frac{4}{5}$ is the same as the fifth part of 4, the fraction $\frac{2}{3}$ is multiplied by 4 (according to No. 8), and divided by 5 (according to No. 9).

13. To *divide* one fraction by another, multiply the numerator of the dividend by the denominator of the divisor, to find the numerator of the quotient; and, to find its denominator, multiply the denominator of the dividend by the numerator of the divisor.[*]

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[*] This rule is usually expressed briefly thus:—Invert the divisor, and proceed as in multiplication.

Thus, if $\frac{3}{4}$ be divided by $\frac{7}{9}$, the quotient is $\frac{27}{28}$. It will be seen that this operation is the same as dividing (No. 9) the dividend by the numerator of the divisor, and multiplying (No. 8) by the denominator. If $\frac{3}{4}$ be divided by 7, the quotient is $\frac{3}{28}$. But, $\frac{7}{9}$ being only a ninth part of 7, the quotient obtained by dividing by $\frac{7}{9}$ will be nine times as great as that resulting from the division by 7: and therefore $\frac{3}{28}$ is multiplied by 9.

14. It follows from the nature of a fraction, that if it be multiplied by its denominator, the product is its numerator. Thus, five times $\frac{3}{5}$ is 3: for $\frac{3}{5}$ means a fifth part of 3; and five times the fifth part of any quantity is the quantity itself.

PROPOSITIONS.

Prop. I. Theor.—If there be four numbers such, that the quotients obtained by dividing the first by the second, and the third by the fourth, are equal; the first has to the second the same ratio that the third has to the fourth.

Let A, B, C, D be four magnitudes, such that $\frac{A}{B}=\frac{C}{D}$: then A : B :: C : D.

For, let m and n be any whole numbers, and multiply (No. 8) the fractions $\frac{A}{B}$ and $\frac{C}{D}$ by m, and (No. 9) divide the product by n; then $\frac{mA}{nB}=\frac{mC}{nD}$. Now, if mA be greater than nB, mC will also be greater than nD; for, if this were not so, $\frac{mA}{nB}$ would not be equal to $\frac{mC}{nD}$. In like manner it might be shown, that if mA be equal to nB, mC will be equal to nD; and that if mA be less than nB, mC will be less than nD. But mA, mC are any like multiples whatever of A, C: and nB, nD any whatever of B, D: and therefore (V. def. 5) A : B :: C : D. Therefore, if there be four numbers, &c.

Schol. This proposition comprehends in it proposition C, of this book, being the same as it, when $\frac{A}{B}$ or $\frac{C}{D}=p$ or $\frac{1}{p}$, p being a whole number.

Cor. If AD = BC, by dividing by B and D (Nos. 4 and 6) we get $\frac{A}{B}=\frac{C}{D}$; and therefore, by this proposition, A : B :: C : D. Hence, if the product of two numbers be equal to that of two others, the one pair may be taken as the extremes and the other as the means of an analogy.

PROP. II. THEOR.—If any four numbers be proportional, and if the first be divided by the second, and the third by the fourth, the quotients are equal.

Let A : B :: C : D; then $\frac{A}{B}=\frac{C}{D}$.

If A and B be whole numbers, let the first and third terms be multiplied by B, and the second and fourth by A, and the products are AB, AB, BC, AD. Now, since the first and second of these are the same, the third and fourth are (V. def. 5) equal; that is, AD = BC; and by dividing these (Nos. 4 and 6) by B and D, we find (V. ax. 2) $\frac{A}{B}=\frac{C}{D}$.

If A and B be fractions, let $A=\frac{E}{m}$, and $B=\frac{F}{n}$, so that (Nos. 3 and 14) $mA=E$, and $nB=F$; the numerators and denominators E, F, m, n being whole numbers. Then (hyp.) $\frac{E}{m} : \frac{F}{n} :: C : D$. Multiply the first and third of these by mF, and the second and fourth by nE, and the products are EF, EF, mFC, and nED. Now, the first and second of these being the same, the third and fourth (V. def. 5) are equal: that is, $n\text{ED}=m\text{FC}$, or $mn\text{AD}=mn\text{BC}$, since $E=mA$, and $F=nB$. Hence, by dividing these (Nos. 4 and 6) by m and n, we get (V. ax. 2) AD = BC; and the rest of the proof is the same as in the first case. Therefore, if any four numbers, &c.*

Schol. 1. This proposition comprehends proposition D, of this book, being the same as it, when $\frac{A}{B}=p$ or $\frac{1}{p}$, p being a whole number.

Schol. 2. From this proposition and the foregoing, it appears, that if two fractions be equal, the numerator of the one is to its denominator, as the numerator of the other to its denominator; and that if the first and second of four proportional numbers be made the numerator and denominator of one fraction, and the third and fourth those of another, the two fractions are equal.

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* If either A or B be a whole number, the proof is included in the second part of the demonstration given above. Thus, if A be a whole number, we have simply E = A and $m=1$, and every thing will proceed as above. The proof would also be readily obtained by substituting for B as before, but retaining A unchanged.

If A and B be incommensurable, such as the numbers expressing the lengths of the diagonal and side of a square, the lengths of the diameter and circumference of a circle, &c., their ratios may be approximated as nearly as we please. Thus, the diagonal of a square is to its side, as $\frac{14}{10}$: 1, nearly; as $\frac{141}{100}$: 1, more nearly; as $\frac{1414}{1000}$: 1, still more nearly, &c. Hence, in such cases we can have no hesitation in admitting the truth of the proposition, as we see that it holds with respect to numbers the ratio of which differs from that of the proposed numbers by a quantity, which may be rendered as small as we please,—smaller in fact than anything that can be assigned.

This is the same in substance as that the two expressions, A : B :: C : D, and $\frac{A}{B}=\frac{C}{D}$, are equivalent, and may be used for one another.

Cor. 1. It appears in the demonstration of this proposition, that AD=BC; that is, if four numbers be proportionals, the product of the extremes is equal to the product of the means. Hence, if the product of the means be divided by one of the extremes, the quotient is the other; and thus we have a proof of the ordinary arithmetical rule for finding a fourth proportional to three given numbers.

Cor. 2. It is evident, that if A be greater than B, C must be greater than D; if equal, equal; and if less, less: as otherwise $\frac{A}{B}$ and $\frac{C}{D}$ could not be equal. This is the same as Prop. A.

Cor. 3. If A : B :: C : D, and consequently $\frac{A}{B}=\frac{C}{D}$, by multiplying these fractions by $\frac{m}{n}$, we get $\frac{mA}{nB}=\frac{mC}{nD}$, or mA : nB :: mC : nD; which is the same as the 4th proposition of this book.

Cor. 4. If A be greater than B, the fraction $\frac{A}{C}$ is evidently greater than $\frac{B}{C}$, and the fraction $\frac{C}{A}$ less than $\frac{C}{B}$: that is, of two unequal numbers, the greater has a greater ratio to a third than the less has; and a third number has a greater ratio to the less than it has to the greater; which is the 8th proposition of this book.

Cor. 5. Conversely, if $\frac{A}{C}$ be greater than $\frac{B}{C}$, A is greater than B; and, if $\frac{C}{A}$ be less than $\frac{C}{B}$, A is also greater than B; which is the 10th proposition of this book.

PROP. III. THEOR. (V. B).—If four numbers be proportionals, they are proportionals also when taken inversely.

If A : B :: C : D: then, inversely, B : A :: D : C.

For (Sup. 2, cor. 1) BC=AD; and hence by dividing (Nos. 4 and 6) by A and C, we obtain $\frac{B}{A}=\frac{D}{C}$, or (Sup. 2, schol. 2) B : A :: D : C. Therefore, if four numbers, &c.

PROP. IV. THEOR. (V. 16).—If four numbers be proportionals, they are also proportionals when taken alternately.

If A : B :: C : D; then, alternately, A : C :: B : D.

For (Sup. 2, cor. 1) AD=BC; whence, by dividing by C and D we get $\frac{A}{C}=\frac{B}{D}$; or (Sup. 2, schol. 2) A : C :: B : D. Therefore, if four numbers, &c.

Schol. When the first and second terms are not of the same kind as the third and fourth, the terms cannot be taken alternately, as ratios would thus be instituted between heterogeneous magnitudes.

Prop. V. Theor. (V. 7).—Equal numbers have the same ratio to the same number; and the same has the same ratio to equal numbers.

Let A and B be equal numbers, and C a third: then A : C :: B : C, and C : A :: C : B.

For, A and B being equal, the fractions $\frac{A}{C}$ and $\frac{B}{C}$ are also equal, or, which is the same, A : C :: B : C: and, by inversion, (Sup. 3) C : A :: C : B. Therefore, equal numbers, &c.

Prop. VI. Theor. (V. 9).—Numbers which have the same ratio to the same number are equal; and those to which the same has the same ratio are equal.

If A : C :: B : C, or if C : A :: C : B, A is equal to B.

For, since $\frac{A}{C}=\frac{B}{C}$, by multiplying by C (No. 14) we get $A=B$. The proof of the second part is the same as this, since, by inversion, (Sup. 3) the second analogy becomes the same as the first. Therefore, numbers, &c.

Prop. VII. Theor. (V. 11.)—Ratios that are equal to the same ratio are equal to one another.

If A : B :: C : D, and E : F :: C : D; then A : B :: E : F.

For, since $\frac{A}{B}=\frac{C}{D}$, and $\frac{E}{F}=\frac{C}{D}$, therefore (I. ax. 1) $\frac{A}{B}=\frac{E}{F}$: that is, (Sup. 2, schol. 2) A : B :: E : F. Therefore, ratios, &c.

Prop. VIII. Theor. (V. 12).—Of numbers which are proportionals, as any one of the antecedents is to its consequent, so are all the antecedents taken together, to all the consequents.

If A : B :: C : D :: E : F; then A : B :: A+C+E : B+D+F.

Since $\frac{A}{B}=\frac{C}{D}=\frac{E}{F}$, put each fraction equal to q, and multiply (No. 14) by the denominators; then $A=Bq$, $C=Dq$, and $E=Fq$. Hence, by addition, $A+C+E=(B+D+F)\,q$; and by dividing by $B+D+F$, we get $q=\frac{A+C+E}{B+D+F}$. But $q=\frac{A}{B}$; and therefore $\frac{A}{B}=\frac{A+C+E}{B+D+F}$, or A : B :: A+C+E : B+D+F. Therefore, &c.

Prop. IX. Theor. (V. 15).—Magnitudes have the same ratio to one another that their like multiples have.

Let A and B be two magnitudes; then, n being a whole number A : B :: nA : nB.

For (No. 10) $\frac{A}{B}=\frac{nA}{nB}$, or A : B :: nA : nB. Therefore, magnitudes, &c.

Prop. X. Theor. (V. 18 and 17).—If four numbers be proportionals; then, (1) by *composition*, the sum of the first and second is to the second, as the sum of the third and fourth to the fourth: and, (2) by *division*, the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.

If A : B :: C : D; then, by composition, A + B : B :: C + D : D; and by division, A — B : B :: C — D : D.

1. Since (hyp.) $\frac{A}{B}=\frac{C}{D}$, and since $\frac{B}{B}=\frac{D}{D}$; add (No. 11) the latter fractions to the former, each to each, and there results $\frac{A+B}{B}=\frac{C+D}{D}$, or A + B : B :: C + D : D.

2. By subtracting (No. 11) the latter pair of the same fractions from the former, each from each, we obtain $\frac{A-B}{B}=\frac{C-D}{D}$; or A — B : B :: C — D : D. If, therefore, &c.

Cor. By dividing the fractions which were found above by addition, by those which were found by subtraction, we get $\frac{A+B}{A-B}=\frac{C+D}{C-D}$; or (Sup. 2, schol. 2) A + B : A — B :: C + D : C — D: that is, if four numbers be proportional, the sum of the first and second terms is to their difference, as the sum of the third and fourth terms is to their difference. It is evident, that if B be greater than A, the analogy would become B + A : B — A :: D + C : D — C.

Prop. XI. Theor. (V. E).—If four numbers be proportional; then, by *conversion*, the first is to its excess above the second, as the third to its excess above the fourth.

If A : B :: C : D; then, by conversion, A : A—B :: C : C — D.

For, since (hyp. and inver.) $\frac{B}{A}=\frac{D}{C}$, and since $\frac{A}{A}=\frac{C}{C}$; take (No. 11) the former fractions from the latter, each from each, and there remains $\frac{A-B}{A}=\frac{C-D}{C}$, or (by inver.) A : A — B :: C : C — D. Therefore, if four numbers, &c.

Prop. XII. Theor. (V. 24).—**If there be numbers forming two or more analogies which have common consequents, the sum of all the first antecedents is to their common consequent, as the sum of all the other antecedents is to their common consequent.**

If A : B :: C : D, and E : B :: F : D; then A + E : B :: C + F : D.

For (hyp.) $\frac{A}{B}=\frac{C}{D}$, and $\frac{E}{B}=\frac{F}{D}$; and hence, by addition, (No. 11) $\frac{A+E}{B}=\frac{C+F}{D}$, or A + E : B :: C + F : D. If, therefore, &c.

Prop. XIII. Theor. (V. 22)—**If there be three or more numbers, and as many others, which, taken two and two in order, have the same ratio; then, *ex æquo*, the first has to the last of the first rank the same ratio that the first has to the last of the second rank.**

If the two ranks of numbers, A, B, C, D, and E, F, G, H, be such that A : B :: E : F, B : C :: F : G, and C : D :: G : H; then A : D :: E : H.

For, since (hyp.) $\frac{A}{B}=\frac{E}{F}$, $\frac{B}{C}=\frac{F}{G}$, and $\frac{C}{D}=\frac{G}{H}$; by multiplying together (Nos. 3 and 12) the first, third, and fifth fractions, and the second, fourth, and sixth, we obtain $\frac{ABC}{BCD}=\frac{EFG}{FGH}$; or, (No. 10) by dividing the terms of the first of these fractions by BC, and those of the second by FG, $\frac{A}{D}=\frac{E}{H}$, or A : D :: E : H. Therefore, if there be three, &c.

This proposition might also be enunciated thus: If there be numbers forming two or more analogies, such that the consequents in each are the antecedents in the one immediately following it, an analogy will be obtained by taking the antecedents of the first analogy, and the consequents to the last for its antecedents and consequents.

Prop. XIV. Theor. (V. 23).—**If there be three or more numbers, and as many others, which, taken two and two in a cross order, have the same ratio; then, *ex æquo inversely*, the first has to the last of the first rank the same ratio which the first has to the last of the second rank.**

If the two ranks of numbers, A, B, C, D, and E, F, G, H, be such that A : B :: G : H, B : C :: F : G, and C : D :: E : F; then, *ex æquo inversely*, A : D :: E : H.

For, since (hyp.) $\frac{A}{B}=\frac{G}{H}$, $\frac{B}{C}=\frac{F}{G}$, and $\frac{C}{D}=\frac{E}{F}$; by multiplying together (Nos. 3 and 12) the fractions as in the preceding proposition, we get $\frac{ABC}{BCD}=\frac{GFE}{HGF}$; whence, by dividing the terms of the first of these fractions by BC, and those of the second by GF, we obtain $\frac{A}{D}=\frac{E}{H}$, or A : D :: E : H. If therefore, &c.

This proposition may also be enunciated thus: If there be numbers forming two or more analogies, such that the means of each are the extremes of the one immediately following it, another analogy may be obtained by taking the extremes of the first analogy, and the means of the last for its extremes and means.

PROP. XV. THEOR.—If there be numbers forming two or more analogies, the products of their corresponding terms are proportionals.

If A : B :: C : D, E : F :: G : H, and K : L :: M : N; then AEK : BFL :: CGM : DHN.

For (hyp.) $\frac{A}{B}=\frac{C}{D}$, $\frac{E}{F}=\frac{G}{H}$, and $\frac{K}{L}=\frac{M}{N}$; and taking (Nos. 3 and 12) the products of the corresponding terms of these fractions, we obtain $\frac{AEK}{BFL}=\frac{CGM}{DHN}$, or AEK : BFL :: CGM : DHN. Therefore, if there be numbers, &c.

Cor. 1. Hence, if there be two analogies consisting of the same terms A, B, C, D, we have $A^2 : B^2 :: C^2 : D^2$; if there be three, we have $A^3 : B^3 :: C^3 : D^3$, &c.: and it thus appears, that like powers of proportional numbers are themselves proportional.

Cor. 2. Like roots of proportional numbers are proportional. Thus, if A : B :: C : D, let $\sqrt{A} : \sqrt{B} :: \sqrt{C} : \sqrt{E}$. Then, by the preceding corollary, A : B :: C : E. But (hyp.) A : B :: C : D; and therefore (Sup. 7) C : E :: C : D, and (Sup. 6) E = D, and consequently $\sqrt{A} : \sqrt{B} :: \sqrt{C} : \sqrt{E}$, or $\sqrt{D}$.

PROP. XVI. THEOR. (V. 25).—The sum of the greatest and least of four proportional numbers is greater than the sum of the other two.

If A : B :: C : D, and if A be the greatest, and therefore (Sup. 2, cor. 2) D the least; A and D are together greater than B and C.

For (by conversion) A : A − B :: C : C − D, and, alternately, A : C :: A − B : C − D. But (hyp.) A > C, and therefore (Sup. 2, cor. 2) A − B > C − D. To each of these add B; then A > B + C − D. Add again, D; then, A + D > B + C. Therefore, &c.

Cor. Hence the mean of three proportional numbers is less than half the sum of the extremes.

PROP. XVII. THEOR.—In numbers which are continual proportionals, the first is to the third as the second power of the first to the second power of the second; the first to the fourth as the third power of the first to the third power of the second; the first to the fifth as the fourth power of the first to the fourth power of the second; and so on.

If A, B, C, D, E, &c. be continual proportionals; $A : C :: A^2 : B^2$; $A : D :: A^3\ B^3$; $A : E :: A^4 : B^4$, &c.

For, since (V. def. 8) $A : B :: B : C$, and since $A : B :: A : B$, we have (Sup. 15) $A^2 : B^2 :: AB : BC$, or, dividing the third and fourth terms by B, $A^2 : B^2 :: A : C$.

Again, since $A^2 : B^2 :: A : C$,
and $A : B :: C : D$, we have (Sup. 15)

$A^3 : B^3 :: AC : CD$, or dividing the third and fourth terms by C, $A^3 : B^3 :: A : D$; and so on, as far as we please. Therefore, &c.

Cor. Hence (V. def. 11 and 12) the ratio which is duplicate of that of any two numbers, is the same as the ratio of their squares; that which is triplicate of their ratio, the same as the ratio of their cubes, &c.

PROP. XVIII. THEOR.—A ratio which is compounded of other ratios, is the same as the ratio of the products of their homologous terms.

Let the ratio of A to D be compounded of the ratios of A to B, B to C, and C to D; the ratio of A to D is the same as that of ABC, the product of the antecedents, to BCD, the product of the consequents.

For, since $A : D :: A : D$, multiply the terms of the second ratio by BC: then (Sup. 9) $A : D :: ABC : BCD$. Therefore, &c.

PROP. XIX. THEOR.—In numbers which are continual proportionals, the difference of the first and second is to the first, as the difference of the first and last is to the sum of all the terms except the last.

If A, B, C, D, E be continual proportionals, $A - B : A :: A - E : A + B + C + D$.

For, since (hyp.) $A : B :: B : C :: C : D :: D : E$, we have (Sup. 8) $A : B :: A + B + C + D : B + C + D + E$. Hence (conv.) $A : A - B :: A + B + C + D : A - E$; and (inver.) $A - B : A :: A - E : A + B + C + D$.

It is evident that if A were the least term, and E the greatest, we should get in a similar manner, $B - A : A :: E - A : A + B + C + D$. Therefore, in numbers, &c.

Cor. If the series be an infinite decreasing one, the last term will vanish, and, if S be put to denote the sum of the series, the analogy will become A—B : A :: A : S; and this, if rA be put instead of B, and the first and second terms be divided by A, will be changed into 1—r : 1 :: A : S. The number r is called the *common ratio*, or *common multiplier*, of the series, as by multiplying any term by it, the succeeding one is obtained.

This Supplement contains all the principal propositions of the fifth book, and some additional ones of importance. The following list shows the propositions of the fifth book contained in this tract.

EUCLID.	SUPPLEMENT.	EUCLID.	SUPPLEMENT.	EUCLID.	SUPPLEMENT.
4	2 cor. 3	15	9	24	12
7	5	16	4	25	16
8	2 cor. 4	17	10	A	2 cor. 2
9	6	18	10	B	3
10	2 cor. 5	19	11 cor.	C	1 schol.
11	7	22	13	D	2 schol. 1
12	8	23	14	E	11

BOOK VI.

DEFINITIONS.

1. *Similar rectilineal figures* are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

2. Two magnitudes are said to be *reciprocally proportional* to two others, when one of the first pair is to one of the second, as the remaining one of the second is to the remaining one of the first.

3. A straight line is said to be cut in *extreme and mean ratio*, when the whole is to one of the segments, as that segment is to the other.

4. The *altitude* of any figure is the straight line drawn from its vertex perpendicular to its base.

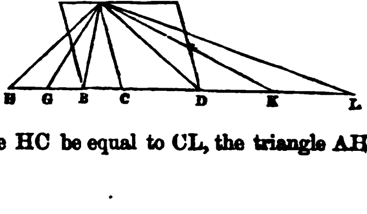
5. A straight line is said to be cut *harmonically*, when it is divided into three segments, such that the whole line is to one of the extreme segments, as the other extreme segment is to the middle one.

PROPOSITIONS.

PROP. I. THEOR.—Triangles and parallelograms of the same altitude are one to another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the same altitude, viz. the perpendicular drawn from the point A to BD: then, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways, and take any number of straight lines BG, GH, each equal to BC; and any number DK, KL, each equal to CD; and join AG, AH, AK, AL. Then, because CB, BG, GH are all equal. the triangles ABC, AGB, AHG are (I. 38) all equal. Therefore, whatever multiple the base HC is of BC, the same multiple is the triangle AHC of ABC. For the same reason, whatever multiple LC is of CD, the same multiple is the triangle ALC of ADC. Also, if the base HC be equal to CL, the triangle AHC



is equal (I. 38) to ALC; and if the base HC be greater than CL, likewise (I. 38, cor. 1) the triangle AHC is greater than ALC; and if less, less. Therefore, since there are four magnitudes, viz. the two bases, BC, CD, and the two triangles ABC, ACD; and of the base BC, and the triangle ABC, the first and third, any like multiples whatever have been taken, viz. the base HC, and the triangle AHC; and of the base CD, and the triangle ACD, the second and fourth, have been taken any like multiples whatever, viz. the base CL, and the triangle ALC; and that it has been shown, that, if the base HC be greater than CL, the triangle AHC is greater than ALC; if equal, equal; and if less, less: therefore (V. def. 5) as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

Again, because (I. 41) the parallelogram CE is double of the triangle ABC, and the parallelogram CF of the triangle ACD, and that (V. 15) magnitudes have the same ratio which their like multiples have; as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF. But it has been shown, that BC is to CD, as the triangle ABC to the triangle ACD; and as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF: therefore (V. 11) as the base BC is to the base CD, so is the parallelogram EC to the parallelogram CF. Wherefore, triangles, &c.*

Cor. 1. From this it is plain, that triangles and parallelograms which have equal altitudes, are one to another as their bases.

Let the figures be placed so as to have their bases in the same straight line; and perpendiculars being drawn from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel (I. 33) to that in which their bases are, because the perpendiculars are both equal and parallel to one another. Then, if the same construction be made as in the proposition, the demonstration will be the same.

Cor. 2. Hence, if A, B, C be any three straight lines, we have A : B :: A.C : B.C.

Cor. 3. So, likewise, if the straight lines A, B, C, D be proportional, and E and F be any other straight lines, we shall have, according to the preceding corollary, and the 11th proposition of the fifth book, A.E : B.E :: C.F : D.F.

Prop. II. Theor.—If a straight line be parallel to the base of a triangle, it cuts the other sides, or those produced, proportionally, and the segments between the parallel and the base are homologous to one another: and (2) if the

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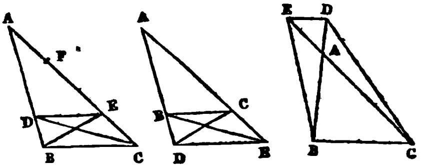
* This proposition may be briefly demonstrated thus: Let a perpendicular drawn from A to BD be called P. Then, ½P.BC will be equal to the area of the triangle ABC, and ½P.CD that of ACD. Dividing, therefore, the former of these equals by the latter, we get $\frac{\frac{1}{2}P.BC}{\frac{1}{2}P.CD}$, or $\frac{BC}{CD}=\frac{ABC}{ACD}$, or (V. Sup. 2, schol. 2) BC : CD :: ABC : ACD. In extending this method of proof to the parallelograms, we have merely to use P instead of ½P.

sides of a triangle, or the sides produced, be cut proportionally, so that the segments between the points of section and the base are homologous to one another, the straight line which joins the points of section is parallel to the base.*

1. Let DE be parallel to BC, one of the sides of the triangle ABC; BD : DA :: CE : EA.



Join BE, CD. Then (I. 37) the triangles BDE, CDE are equal, because they are on the same base DE, and between the same parallels, DE, BC. Now ADE is another triangle, and (V. 7) equal magnitudes have to the same, the same ratio; therefore, as the triangle BDE to ADE, so is the triangle CDE to ADE. But

(VI. 1) as the triangle BDE to ADE, so is BD to DA;

because, having the same altitude, viz. the perpendicular drawn from E to AB, they are to one another as their bases; and for the same reason,

as the triangle CDE to ADE, so is CE, to EA.

Therefore (V. 11) as BD : DA :: CE : EA.

2. Next, let the sides AB, AC of the triangle ABC, or those produced, be cut proportionally in the points D, E; that is, so that BD : DA :: CE : EA, and join DE; DE is parallel to BC.

The same construction being made, because (hyp.)

as BD : DA :: CE : EA; and (VI. 1)

as BD to DA, so is the triangle BDE to the triangle ADE; and as CE to EA, so is the triangle CDE to ADE: therefore (V. 11) the triangle BDE is to ADE, as the triangle CDE to ADE; that is, the triangles BDE, CDE have the same ratio to ADE; and therefore (V. 9) the triangles BDE, CDE are equal; and they are on the same base DE, and on the same side of it: therefore (I. 39) DE is parallel to BC. Wherefore, if a straight line, &c.

Cor. The triangles which two intersecting straight lines form with two parallel ones, have their sides which are on the intersecting lines proportional; and those sides are homologous which are in the same straight line: and, (2) conversely, if two straight lines form with two intersecting ones, triangles which have their sides that are on the intersecting lines proportional, the sides which are in the same straight line with one another being homologous, those straight lines are parallel.

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* The enunciation of this proposition which is given by Dr. Simson and others, is defective, and might lead to error in its application, as it does not point out what lines are homologous to one another in the analogies.

It is plain, that, instead of one proposition, this is in reality two, which are converses of one another. The same is the case in the third, fourteenth, fifteenth, sixteenth, seventeenth, and twenty-second of this book.

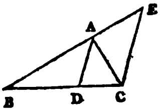
1. Let DE and BC (first and third figures) be the parallels, and let them be cut by the straight lines BD, CE, which intersect each other in A; then BA : AC :: DA : AE. For, since BD : DA :: CE : EA, we have, by composition in the first figure, and, by division in the third, BA : DA :: CA : EA, and, alternately, BA : AC :: DA : AE.

2. But if BA : AC :: DA : AE, DE and BC are parallel. For, alternately BA : DA :: CA : EA; then, in the first figure by division, and in the third by composition, we have BD : DA :: CE : EA; and therefore, by the second part of this proposition, DE is parallel to DC.

PROP. III. THEOR.—The straight line which bisects an angle of a triangle, divides the opposite side into segments which have the same ratio to one another as the adjacent sides of the triangle have: and (2) if the segments of the base have the same ratio as the adjacent sides, the straight line drawn from the vertex to the point of section, bisects the vertical angle.

1. Let the angle BAC of the triangle ABC be bisected by the straight line AD; then BD : DC :: BA : AC.

Through C draw (I. 31) CE parallel to DA; then (I. 29, cor. 1) BA produced will meet CE; let them meet in E. Because AC meets the parallels AD, EC, the angle ACE is equal (I. 29) to the alternate angle CAD; and because BAE meets the same parallels, the angle E is equal (I. 29, part 2) to BAD: therefore (I. ax. 1) the angles ACE, AEC are equal, because they are respectively equal to the equal angles, DAC, DAB; and consequently AE is equal (I. 6) to AC. Now (VI. 2) because AD is parallel to EC, one of the sides of the triangle BCE, BD : DC :: BA : AE; but AE is equal to AC; therefore (V. 7) BD : DC :: BA : AC.



2. Let now BD : DC :: BA : AC, and join AD; the angle BAC is bisected by AD.

The same construction being made, because

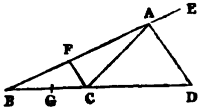
(hyp.) BD : DC :: BA : AC; and
(VI. 2) BD : DC :: BA : AE,

since AD is parallel to EC; therefore (V. 11) BA : AC :: BA : AE: consequently (V. 9) AC is equal to AE; and (I. 5) the angles AEC, ACE are therefore equal. But (I. 29) the angle BAD is equal to E, and DAC to ACE; wherefore also BAD is equal (I. ax. 1) to DAC; and therefore the angle BAC is bisected by AD. The straight line, therefore, &c.

PROP. A. THEOR.[*]—**If an exterior angle of a triangle be bisected by a straight line which also cuts the base produced,[†] the segments between the bisecting line and the extremities of the base, have the same ratio to one another, as the other sides of the triangle have: and (2) if the segments of the base produced, have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section bisects the exterior angle of the triangle.**

1. Let an exterior angle CAE of any triangle ABC be bisected by AD which meets the opposite side produced in D; then BD : DC :: BA : AC.

Through C draw (I. 31) CF parallel to AD; and because AC meets the parallels AD, FC, the angle ACF is equal (I. 29) to CAD; and because the straight line FAE meets the parallels AD, FC, the angle CFA is equal to DAE; therefore also ACF, CFA are (I. ax. 1) equal to one another, because they are respectively equal to the equal angles DAC, DAE; and consequently (I. 6) AF is equal to AC. Then (VI. 2) because AD is parallel to FC, a side of the triangle BCF, BD : DC :: BA : AF; but AF is equal to AC; as therefore BD : DC :: BA : AC.



2. Let now BD : DC :: BA : AC, and join AD; the angle CAD is equal to DAE.

The same construction being made, because BD : DC :: BA : AC; and that (VI. 2) BD : DC :: BA : AF; therefore (V. 11) BA : AC :: BA : AF; wherefore (V. 9) AC is equal to AF, and (I. 5) the angle AFC to ACF. But (I. 29) the angle AFC is equal to EAD, and ACF to CAD; therefore also (I. ax. 1) EAD is equal to CAD. Wherefore, &c.

Cor. If G be the point in which BC is cut by the straight line bisecting the angle BAC,

we have (VI. 3) BG : GC :: BA : AC;

and by this proposition, BD : DC :: BA : AC;

whence, (V. 11) BD : DC :: BG : GC, and therefore (VI. def. 5) BD is divided harmonically in G and C.

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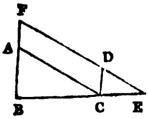
[*] This proposition was inserted in the elements by Dr. Simson; who, besides stating that it is merely a second case of the third proposition, remarks, that "the demonstration of it is very like to that of the first case, and upon this account may, probably, have been left out, as also the enunciation, by some unskilful editor." Elrington conceives this view to be corroborated by the expression, "which also cuts the base," in the enunciation; as this expression is necessary only in relation to the line which bisects the exterior angle. The enunciations of both cases may be easily combined in one. For the sake of simplicity, however, they are here allowed to continue separate.

[†] If the triangle be isosceles, the line bisecting the exterior angle at the vertex is parallel to the base. In this case, the segments may be regarded as infinite, and therefore equal, their difference, the base being infinitely small in comparison of them.

Prop. IV. Theor.[*] — **The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.**

Let ABC, DCE be equiangular triangles, having the angle ABC equal to DCE, and ACB to DEC, and consequently (I. 32, cor. 5) BAC equal to CDE: the sides about the equal angles are proportionals; and those are the homologous sides which are opposite to the equal angles.

Let the triangles be placed on the same side of a straight line BE, so that sides, BC, CE, which are opposite to equal angles, may be in that straight line and contiguous to one another; and so that neither the equal angles ABC, DCE, nor ACB, DEC at the extremities of those sides may be adjacent. Then, because (I. 17) the angles ABC, ACB are together less than two right angles, ABC and DEC, which (hyp.) is equal to ACB, are also less than two right angles; wherefore (I. ax. 12) BA, ED will meet, if produced: let them be produced and meet in F. Again, because the angle ABC is equal to DCE, BF is parallel (I. 28) to CD; and, because the angle ACB is equal to DEC, AC is parallel to FE. Therefore FACD is (I. def. 24) a parallelogram; and consequently (I. 34) AF is equal to CD, and AC to FD. Now (VI. 2) because AC is parallel to FE, one of the sides of the triangle FBE,



BA : AF :: BC : CE.

But AF is equal to CD; therefore (V. 7)

as BA : CD :: BC : CE,

and alternately (V. 16) as AB : BC :: DC : CE.

Again (VI. 2), because CD is parallel to BF, as

BC : CE :: FD : DE; but FD is equal to AC; therefore,

as BC : CE :: AC : DE; and, alternately,

as BC : CA :: CE : ED.

Therefore, because it has been proved that

AB : BC :: DC : CE, and as BC : CA :: CE : ED;[†]

ex æquo (V. 22), BA : AC :: CD : DE. Therefore, the sides, &c.

Schol. Hence (VI. def. 1) equiangular triangles are similar.

Cor. If two angles of one triangle be respectively equal to two angles of another, their sides are proportional, and the sides opposite to equal angles are homologous. For (I. 32, cor. 5) the remaining angles are equal, and therefore the triangles are equiangular.

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[*] This important proposition is a generalization of the 26th of the first book, as the fifth and sixth of this book are of the 8th and 4th of the first.

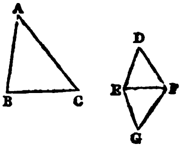
[†] Here, the two ranks of magnitudes are AB, BC, CA, and DC, CE, ED.

PROP. V. THEOR.—If the sides of two triangles, about each of their angles, be proportionals, the triangles are equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF, have their sides proportionals, so that AB : BC :: DE : EF; and BC : CA :: EF : FD; and consequently, ex æquo, BA : AC :: ED : DF; the triangles are equiangular, and the equal angles are opposite to the homologous sides, viz. the angle ABC equal to DEF, BCA to EFD, and BAC to EDF.

At the points E, F, in the straight line EF, make (I. 23) the angle FEG equal to B, and EFG equal to C. Then (VI. 4, cor.) the triangles ABC, GEF have their sides opposite to the equal angles proportionals; wherefore

AB : BC :: GE : EF; but (hyp.)
AB : BC :: DE : EF.

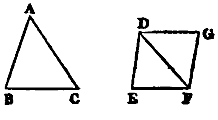


Therefore (V. 11) DE : EF :: GE : EF; whence, since DE and GE have the same ratio to EF, they are (V. 9) equal. It may be shown in a similar manner that DF is equal to FG; and because, in the triangles DEF, GEF, DE is equal to EG, EF common, and DF equal to GF; therefore (I. 8) the angle DEF is equal to GEF, DFE to GFE, and EDF to EGF. Then, because the angle DEF is equal to GEF, and (const.) GEF to ABC; therefore the angle ABC is equal to DEF. For the same reason, ACB is equal to DFE, and A to D. Therefore the triangles ABC, DEF are equiangular: wherefore, if the sides, &c.

PROP. VI. THEOR.—If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals; the remaining angles are equal, each to each, viz. those which are opposite to the homologous sides.

Let the triangles ABC, DEF have the angles BAC, EDF equal, and the sides about those angles proportionals; that is, BA : AC :: ED : DF; the angle ABC is equal to DEF, and ACB to DFE.

Make (I. 23) the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to ACB. Then (VI. 4, cor.)



BA : AC :: GD : DF. But (hyp.)
BA : AC :: ED : DF;

and therefore (V. 11) ED : DF :: GD : DF; wherefore ED is equal (V. 9) to DG. Now DF is common to the two triangles EDF, GDF; and the angle EDF is equal (const.) to

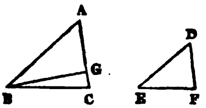
GDF; therefore the angle DFG is equal (I. 4, part 3) to DFE, and G to E. But (const.) the angle DFG is equal to ACB: therefore ACB is equal to DFE; and (hyp.) the angle BAC is equal to EDF; wherefore also (I. 32, cor. 5) the remaining angles B and E are equal. Therefore, if two triangles, &c.*

Prop. VII. Theor.—If two triangles have two sides of the one proportional to two sides of the other; and if the angles opposite to one pair of the homologous sides be equal, and those opposite to the other pair be either both acute, or not acute; the angles contained by the proportional sides are equal.

In the triangles ABC, DEF, let AB : BC :: DE : EF, and let the angles A and D be equal: then if C and F be either both acute, or not acute, the angles ABC, DEF are equal.

For, if they be not equal, let ABC be the greater, and (I. 23) make the angle ABG equal to DEF: then, since (hyp.) A and D are equal, AGB is equal (I. 32, cor. 5) to DFE; and (VI. 4).

AB : BG :: DE : EF; but (hyp.)
AB : BC :: DE : EF;



therefore (V. 11) AB : BG :: AB : BC. Hence (V. 9) BG and BC are equal, because AB has the same ratio to each of them; and therefore (I. 5) the angle BGC is equal to C. The angles at G therefore are both acute, or not acute, according as C and F are acute, or not acute.

In the first case, then, in which C and F are both acute, the angles BGA, BGC are together less than two right angles, which (I. 13) is impossible; and in the second case, in which C and F are not acute, the angles BGC, BCG of the triangle CBG are not together less than two right angles, which (I. 17) is impossible. The angle ABC, therefore, is not unequal to DEF; that is, it is equal to it: wherefore, if two triangles, &c.†

Prop. VIII. Theor.—In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the hypotenuse BC: the triangles ADB, ADC are similar to the whole triangle ABC, and to one another.

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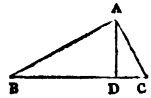
* This proposition may be very easily proved indirectly.

† The proof of this proposition which is given above, is much shorter than that of Euclid. The expression of the enunciation is also altered, with a view to render it more simple and plain.

Because the angle BAC is equal (I. ax. 11) to ADB, each of them being a right angle, and that the angle B is common to the two triangles ABC, ABD; the remaining angle C is equal (I. 32, cor. 5) to the remaining angle BAD. Therefore the triangles ABC, ABD are equiangular and (VI. 4) the sides about their equal angles are proportionals; wherefore (VI. def. 1) the triangles are similar. In the same manner it might be demonstrated, that the triangle ADC is equiangular and similar to ABC: and the triangles ADB, ADC, being each equiangular to ABC, are (I. ax. 1) equiangular, and therefore (VI. 4, and def. 1) similar to each other. Therefore, &c.



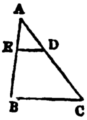
Cor. From this it is manifest, that the perpendicular drawn from the right angle of a right-angled triangle to the hypotenuse, is a mean proportional (V. def. 9) between the segments of the hypotenuse: and also that each of the sides is a mean proportional between the hypotenuse and its segment adjacent to that side. For, (VI. 4) in the triangles BDA, ADC,

as BD : DA :: DA : DC; in the triangles ABC, DBA,
as BC : BA :: BA : BD; and, in the triangles ABC, ACD,
as BC : CA :: CA : CD.

Prop. IX. Prob.—From a given straight line to cut off any assigned part, or submultiple.[*]

Let AB be the given straight line; it is required to cut off any assigned part from it.

Draw a straight line AC making any angle with AB; in AC take any point D, and take AC the same multiple of AD, that AB is of the part which is to be cut off from it; join BC, and (I. 31) draw DE parallel to it: AE is the part required.



Because ED is parallel to BC, one of the sides of the triangle ABC, CD : DA :: BE : EA (VI. 2); and, by composition, (V. 18) CA : AD :: BA : AE. But (const.) CA is a multiple of AD; therefore (V. D) BA is the same multiple of AE. Whatever part, therefore, AD is of AC, the same part is AE of AB. Therefore, from the straight line AB the part required is cut off: which was to be done.

Prop. X. Prob.—To divide a given straight line similarly to a given divided straight line; that is, into parts proportional to the parts of the given divided line.

Let AB be the straight line given to be divided, and AC the given line divided in D and E; it is required to divide AB similarly to AC.

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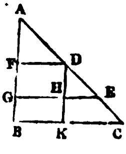
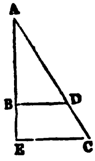
[*] This problem is the same in substance, as to divide a given straight line into any proposed number of equal parts; a method of doing which has already been given in the second corollary to the 34th proposition of the first book.

Let AB, AC be placed so as to contain any angle; join BC, and, (I. 31) through the points D, E, draw DF, EG parallels to it: AB is divided in F and G similarly to AC.

Through D draw DHK parallel to AB. Then each of the figures FH, HB is (I. def. 24) a parallelogram; wherefore DH is equal (I. 34) to FG, and HK to GB. And (VI. 2) because HE is parallel to KC, one of the sides of the triangle DKC, as CE : ED :: KH : HD; but KH is equal to BG, and HD to GF; therefore, as CE : ED :: BG : GF. Again, (VI. 2) because FD is parallel to EG, as ED : DA :: GF : FA. But it has been proved that CE : ED :: BG : GF; and now that ED : DA :: GF : FA; therefore the given straight line AB is divided similarly to AC: which was to be done.



Schol. In a similar manner we may produce a given straight line, so that the whole line so produced may have to the part produced the ratio of two given straight lines. Thus, if AB be the line to be produced, make at A an angle of any magnitude, and take AC and CD equal to the other given lines; join BD, and draw CE parallel to it. Then, since EC is parallel to BD, a side of the triangle ABD, we have (VI. 2) AE : EB :: AC : CD, so that AE has to EB the given ratio.

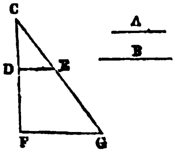


Prop. XI. Prob.—To find a third proportional to two given straight lines.

Let A and B be two given straight lines; it is required to find a third proportional to them.

Take two straight lines CF, CG, containing any angle C; and upon these make CD equal to A, and DF, CE each equal to B. Join DE, and (I. 31) draw FG parallel to it. EG is the third proportional required.

For (VI. 2) since DE is parallel to FG, CD : DF :: CE : EG. But (const.) CD is equal to A, and DF, CE each equal to B: therefore A : B :: B : EG; wherefore to A and B, the third proportional EG is found, which was to be done.*



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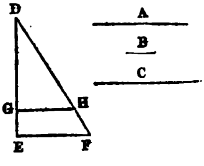
* Other modes of solving this problem may sometimes be employed with advantage. The following are among the most useful:

1. Draw AD (fig. to prop. 8) perpendicular to the indefinite straight line BC, and make DB, DA equal to the given lines: join AB, and draw AC perpendicular to it: DC is the third proportional required. For (VI. 8, cor.) BD : DA :: DA : DC.

2. Draw BC perpendicular to AB, and having made BA and AC equal to the less and greater of the given lines, draw CD and BE perpendicular to AC: AD will be a third proportional to AB and AC, and AE to AC and AB. This follows

Prop. XII. Prob. — To find a fourth proportional to three given straight lines.

Let A, B, C be three given straight lines; it is required to find a fourth proportional to them.



Take two straight lines DE, DF, containing any angle EDF, and make DG equal to A, GE* equal to B, and DH equal to C; and having joined GH, draw (I. 31) EF parallel to it through the point E: HF is the fourth proportional required.

For (VI. 2) since GH is parallel to EF, as DG : GE :: DH : HF; but DG is equal to A, GE to B, and DH to C; therefore as A : B :: C : HF; wherefore, to the three given straight lines, A, B, C, the fourth proportional HF is found; which was to be done.†

Prop. XIII. Prob.—To find a mean proportional between two given straight lines.

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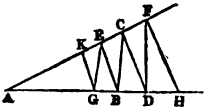
from the fourth proposition of this book, since the triangles ABC, ACD are equiangular, as are also ABC, AEB.



The angles ABC, ACD, &c. are here made right angles. They may be of any magnitude, however, provided they be equal. It is sufficient, therefore, to draw two straight lines, AB, AC, making any angle; to cut off AB, AC equal to the given proportionals; and then, BC being joined, to make the angle ACD equal to ABC, and to draw BE parallel to CD; or to make the angle ABE equal to ACB, and to draw CD parallel to BE.

This method affords an easy means of continuing a series of lines in continual proportion, both ways, when any two successive terms are given. Thus, after CD and BE are drawn, it is only necessary to draw DF, EG, &c. parallel to BC, and FH, GK, &c. parallel to CD; as AD, AF, AH, &c. will be the succeeding terms of the ascending series, and AE, AG, AK, &c. those of the descending one.

* It is plain that GE may be taken on either side of G.

† The solution of this problem may also be effected in several different ways; some of which may be merely indicated to the student, as the proofs present no difficulty; and it is evident that, with slight modification, they are applicable in solving the 11th proposition, which is only a particular case of the 12th.

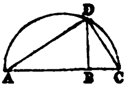
1. Let AE, EC (last fig. to III. 35, page 78) be the second and third terms, placed contiguous, and in the same straight line, and draw BE, making any angle with AC, and equal to the first term; through the three points, A, B, C, describe a circle cutting BE produced in D: ED is the fourth proportional. If AB and CD be joined, the proof will be obtained by means of the triangles ABE, CDE, which are similar.

2. Draw AB, AC (fig. to III. 36, cor. page 80) making any angle, and make AB, AE equal to the second and third terms; then if AC be taken equal to the first term, and a circle be described passing through B, C, and E, and meeting AC in F, AF will be the required line. If BF and EC be joined, the triangles ABF and ACE are similar, and hence the proof is immediately obtained.

3. Make BD and DA (fig. to VI. 8, page 134) perpendicular to each other, and equal to the first and second terms; join AB, and draw AC perpendicular to it; in DA, produced through A, if necessary, take a line equal to the third term, through the upper extremity of which draw a line parallel to AC: the line intercepted on DC, produced if necessary, between D and this parallel, is the fourth proportional required.

Let AB, BC be two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line, and upon AC as diameter describe the semicircle ADC; from B (I. 11) draw BD at right angles to AC : BD is the mean proportional between AB and BC.



Join AD, DC. Then, because the angle ADC in a semicircle is (III. 31) a right angle, and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to AC, DB is a mean proportional (VI. 8, cor.) between AB, BC the segments of the base. Therefore between AB, BC, the mean proportional DB is found: which was to be done.*

Prop. XIV. Theor.—Equal parallelograms which have an angle of the one equal to an angle of the other, have their sides about those angles reciprocally proportional: and (2) parallelograms which have an angle of the one equal to an angle of the other, and the sides about those angles reciprocally proportional, are equal to one another.

1. Let AB, BC be equal parallelograms, which have the angles at B equal; the sides about those angles are reciprocally proportional; that is, DB : BE :: GB : BF.

Let the sides DB, BE be placed in the same straight line, and contiguous, and let the parallelograms be on opposite sides of DE; then (I. 14, cor.) because the angles at B are equal, FB, BG are in one straight line. Complete the parallelogram FE,† and (V.

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4. If any number of antecedents, and the consequent of one of them be given, the other consequents may be readily found on the principle employed in the tenth proposition of this book. Thus, if AD (fig. to that proposition, page 135) be the given consequent and AF its antecedent; then, if FG, GB, &c. be the other antecedents, the segments DE, EC, &c. made on the other line by the parallels, are the remaining consequents.

* Out of several additional ways of solving this problem, the following may be mentioned.

1. On the greater extreme as diameter describe a semicircle; from the diameter cut off a segment equal to the less extreme, through the extremity of which draw a perpendicular cutting the circumference; and the chord drawn from that intersection to the extremity of the diameter common to the two extremes is the required mean. The proof is manifest from III. 31, and VI. 8, cor.

2. Make AD, DC (2d fig. to III. 36, page 79) equal to the given extremes; on AC as chord describe any circle, and a tangent drawn from D will be the required mean. The proof of this is obtained by joining AB and BC, as the triangles ADB, BDC are similar.

When one mean is determined, others may be found between it and the given extremes, and thus three means will be inserted between the given lines; and by finding means between each successive pair of the five terms of which the series then consists, the number of means will be increased to seven. By continuing the process we may find fifteen means, thirty-one means, or any number which is less by one than a power of 2. We cannot find, however, by elementary geometry, any other number of means, such as two, four, or five.

† If AD, CG were produced to meet, it would be easy to show, that AB and BC would be the complements of the parallelograms about the diagonal of the whole parallelogram AC.

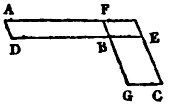
In the demonstration it should in strictness be proved that AF and CE meet when produced. This follows from I. 29, cor. 1.

7) because AB is equal to BC, and that FE is another parallelogram,

AB : FE :: BC : FE. But (VI. 1)

as AB to FE, so is the base DB to BE; and

as BC to FE, so is the base GB to BF;



therefore (V. 11) as DB : BE :: GB : BF. The sides, therefore, of the parallelograms AB, BC, about their equal angles, are (VI. def. 2) reciprocally proportional.

2. But let the sides about the equal angles be reciprocally proportional, viz. DB : BE :: GB : BF; the parallelograms AB, BC are equal.

The same construction being made, because,

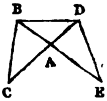
as DB : BE :: GB : BF; and (VI. 1)

as DB : BE :: AB : FE; and as GB : BF :: BC : FE;

therefore (V. 11) as AB : FE :: BC : FE; wherefore (V. 9) the parallelogram AB is equal to the parallelogram BC. Therefore equal parallelograms, &c.

Prop. XV. Theor.—Equal triangles which have an angle of the one equal to an angle of the other, have their sides about those angles reciprocally proportional: and (2) triangles which have an angle of the one equal to an angle of the other, and the sides about those angles reciprocally proportional, are equal to one another.

1. Let ABC, ADE be equal triangles, which have the angles BAC, DAE equal; the sides about those angles are reciprocally proportional; that is, CA : AD :: EA : AB.



Let CA, AD be placed in one straight line, and contiguous, and let the triangles be on opposite sides of CD; then (I. 14, cor.) BA, AE are in one straight line. Join BD; and because the triangles ABC, ADE are equal, and ABD is another triangle; therefore (V. 7)

as CAB is to BAD, so is EAD to BAD. But (VI. 1) as the triangle CAB to BAD, so is CA to AD; and as the triangle EAD to BAD, so is EA to AB; as, therefore, (V. 11) CA : AD :: EA : AB; wherefore (VI. def. 2) the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.

2. But let the sides of the triangles ABC, ADE, about the equal angles be reciprocally proportional, viz. CA : AD :: EA : AB; the triangles ABC, ADE are equal.

The same construction being made,

because CA : AD :: EA : AB; and (VI. 1)

as CA to AD, so is the triangle ABC to BAD; and

as EA to AB, so is the triangle EAD to BAD:

therefore (V. 11) as the triangle BAC to BAD, so is the triangle EAD to BAD; that is, the triangles BAC, EAD have the same ratio to BAD: wherefore (V. 9) the triangles ABC, ADE are equal.* Therefore, equal triangles, &c.

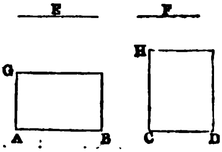
OTHERWISE:—1. Let CA, AD be placed in one straight line and contiguous, and let the triangles be on opposite sides of CAD; then (I. 14, cor.) BA and AE are also in one straight line. Join BD and CE; and, because the triangles ABC, ADE are equal, if BAD be added to each, the whole BCD is equal (I. ax. 2) to the whole BED; and, therefore (I. 39) BD is parallel to CE, and (VI. 2, cor.) CA : AD :: EA : AB.

2. The same construction being made, since CA : AD :: EA : AB, BD is parallel (VI. 2, cor.) to CE; and (I. 37) the triangles BCD, BED are equal to one another: from each of them take ABD, and the remaining triangles ABC, ADE are equal.†

PROP. XVI. THEOR.—If four straight lines be proportionals, the rectangle contained by the extremes is equal to that contained by the means: and (2) if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals.

1. Let AB : CD :: E : F; the rectangle AB.F is equal to the rectangle CD.E.

Draw (I. 11) AG, CH perpendicular to AB, CD; make AG equal to F, and CH equal to E; and complete the parallelograms BG, DH. Because AB : CD :: E : F; and that E is equal to CH, and F to AG; therefore, (V. 7) AB : CD :: CH : AG. Hence the sides of the parallelograms BG, DH, about the equal angles are reciprocally proportional: wherefore (VI. 14) the parallelogram BG is equal to DH. Now, the parallelogram BG is contained by AB and F, because AG is equal to F; and DH is contained by CD and E, because CH is equal to E. Therefore the rectangle AB.F is equal to the rectangle CD.E.



2. If the rectangle AB.F be equal to the rectangle CD.E; then AB : CD :: E : F.

The same construction being made, because the rectangle AB.F is equal to the rectangle CD.E, and that the rectangle BG is contained by AB and F, because AG is equal to F; and the rectangle

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* It is evident from the second corollary to the thirty-eighth proposition of the first book, that this proposition is true as well when the angles are supplemental, as when they are equal.

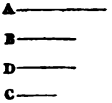
† From either this very easy method of demonstrating the 15th proposition, or from the method given in the text, the proof of the 14th would follow immediately, by means of the 34th of the first book, if the diagonals of the parallelograms were drawn. It may be remarked, that a proof similar to the second one given above, would be obtained by placing the triangles so as to have their equal angles *coinciding*, instead of being vertically opposite. In the figure, the straight line CE, which is not actually drawn, may be conceived to be drawn.

DH by CD and E, because CH is equal to E; therefore the parallelograms BG, DH are equal; and they are equiangular. But (VI. 14) the sides about the equal angles of equal parallelograms are reciprocally proportional: wherefore, as AB : CD :: CH : AG; or (const.) AB : CD :: E : F.* Therefore, &c.

PROP. XVII. THEOR. If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and (2) if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let the three straight lines A, B, C be proportionals, viz. A : B :: B : C; the rectangle A.C is equal to the square of B.

1. Take D equal to B; and because A : B :: B : C, and that B is equal to D; therefore A : B :: D : C. But (VI. 16) if four straight lines be proportionals, the rectangle under the extremes is equal to that under the means: therefore the rectangle A.C is equal to the rectangle B.D. But the rectangle B.D is the square of B; because B is equal to D: therefore the rectangle A.C is equal to the square of B.



2. If the rectangle A.C be equal to the square of B; A : B :: B : C.

The same construction being made, because the rectangle A.C is equal to the square of B, and the square of B is equal to the rectangle B.D, because B is equal to D; therefore the rectangle A.C is equal to the rectangle B.D. But (VI. 16) if the rectangle of the extremes be equal to that of the means, the straight lines are proportionals: therefore A : B :: D : C; but B is equal to D; wherefore, as A : B :: B : C. Therefore, if three straight lines, &c.

PROP. XVIII. PROB.—Upon a given straight line to describe a figure similar to a given rectilineal figure, and such that the given line shall be homologous to an assigned side of the given figure.†

Let AB be the given straight line, on which it is required to describe a rectilineal figure similar to a given rectilineal figure, and such that AB may be homologous to CD, a side of the given figure.

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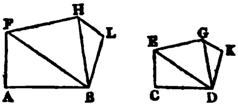
* This proposition, of which the one following it is a case, affords the means of deriving the equality of rectangles, and the proportion of straight lines containing them, from one another. It evidently corresponds to V. Sup. prop. 1. cor. and prop. 2, cor, 1, and it might be regarded as an immediate result of those corollaries, without any distinct proof, if the lines were expressed (I. 46, cor. 4) by lineal units, and the rectangles by superficial ones. This proposition and the fourth of this book, when employed in connexion with one another, form one of the most powerful instruments in geometrical investigations: and they facilitate in a peculiar degree the application of algebra to such inquiries.

† The latter condition may be also expressed by saying, that the two figures are to be "similarly situated" on the two straight lines AB and CD.

First, let the given rectilineal figure be the triangle CDE. Make the angles BAF, ABF respectively equal to DCE, CDE; and (VI. 4, cor.) the triangle ABF is similar to CDE, and has AB homologous to CD.



Again, let the given figure be the quadrilateral CDGE. Join DE, and, as in the first part, describe the triangle ABF having the angles BAF, ABF respectively equal to DCE, CDE, and also the triangle BFH having the angles FBH, BFH respectively equal to EDG, DEG. Then (I. 32, cor. 5) the angles BHF, DGE are equal, and (const.) A and C are equal. Also since (const.) ABF, FBH are respectively equal to CDE, EDG, the whole ABH is equal to the whole CDG. For the same reason, AFH is equal to CEG; and therefore the quadrilateral figures ABHF, CDGE are equiangular. But likewise these figures have their sides about the equal angles proportional. For the triangles ABF, CDE being equiangular, and also BFH, DEG; as BA : AF :: DC : CE; and as FH : HB :: EG : GD. Also, in the same triangles, as AF : FB :: CE : ED; and as FB : FH :: ED : EG; therefore, *ex æquo*, AF : FH :: CE : EG. In the same manner it may be proved, that AB : BH :: CD : DG: wherefore (VI. def. 1) the figures ABHF, CDGE are similar to one another.

Next, let the given figure be CDKGE. Join DG; and, as in the second case, describe the figure ABHF, similar to CDGE, and similarly situated: also, describe the triangle BHL having the angles BHL, HBL respectively equal to DGK, GDK. Then (I. ax. 2) the whole angles FHL, ABL are respectively equal to the whole angles EGK, CDK; and (I. 32, cor. 5) the angles L and K are equal. Therefore the figures ABLHF, CDKGE are equiangular. Again, because the quadrilaterals ABHF, CDGE, and the triangles BLH, DKG are similar; as FH : HB :: EG : GD, and HB : HL :: GD : GK; therefore, *ex æquo*, FH : HL :: EG : GK. In like manner, it may be shown, that AB : BL :: CD : DK : and because the quadrilaterals ABHF, CDGE, and the triangles BLH, DKG are similar, the sides about the angles A and C, L and K, AFH and CEG are proportional. Therefore (VI. def. 1) the five-sided figures ABLHF, CDKGE are similar, and the sides AB and CD are homologous. In the same manner, a rectilineal figure of six or more sides may be described on a given straight line, similar to one given: which was to be done.*

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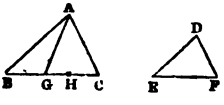
* In practice, if AB be parallel to CD, the construction is most easily effected by drawing AF and BF parallel to CE and DE; then FH and BH parallel to EG and DG; and, lastly, HL and BL parallel to GK and DK. The doing of this is much facilitated by employing the useful instrument, the parallel ruler. For the easiest methods, however, of performing this and many other problems, the student

Prop. XIX. Theor.—Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles, having the angles B and E equal, and AB : BC :: DE : EF, so that (V. def. 13) the side BC is homologous to EF. The triangle ABC has to the triangle DEF, the duplicate ratio of that which BC has to EF.

Take (VI. 11) BG a third proportional to BC, EF, so that BC : EF :: EF : BG, and join GA.*



Then, because
AB : BC :: DE : EF; alternately,
AB : DE :: BC : EF; but (const.)
as BC : EF :: EF : BG;
therefore (V. 11) as AB : DE :: EF : BG. The sides, therefore, of the triangles ABG, DEF, which are about the equal angles, are reciprocally proportional; and therefore (VI. 15) the triangles ABG, DEF are equal. Again, because BC : EF :: EF : BG; and that if three straight lines be proportionals, the first is said (V. def. 11) to have to the third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF. But (VI. 1) as BC to BG, so is the triangle ABC to ABG. Therefore (V. 11) the triangle ABC has to ABG the duplicate ratio of that which BC has to EF. But the triangle ABG is equal to DEF; wherefore also the triangle ABC has to DEF the duplicate ratio of that which BC has to EF.† Therefore, &c.

Cor. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar triangle, similarly described on the second.

Prop. XX. Theor.—Similar polygons may be divided into the same number of similar triangles: (2) these triangles have the same ratio to one another that the polygons have: and (3) the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let AD, FK be similar polygons, and let the sides AB, FG be homologous: the polygons may be divided into the same number of similar triangles: (2) these triangles have, each to each, the

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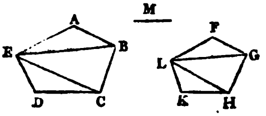
must have recourse to works that treat expressly on such subjects, particularly treatises on practical geometry, surveying, and the use of mathematical instruments.

* The third proportional might be taken to EF and BC, and placed from E along EF produced; and then a triangle equal to ABC would be formed by joining D with the extremity of the produced line.

† This proposition might also be proved by making BH equal to EF, joining AH, and drawing through H a parallel to AC. The triangle cut off by the parallel is equal (I. 26) to DEF. But (VI. 1) the triangle ABC is to the triangle ABH, as BC to BH; and, for the same reason, the triangle ABH is to the triangle cut off by the parallel, or to DEF, as BC to BH, or (const.) as BH to BG. Therefore, *ex æquo*, the triangle ABC has to the triangle DEF, the same ratio that BC has to BG, or (V. def. 11) the duplicate ratio of that which BC has to EF.

same ratio which the polygons have: and (3) the polygon AD has to the polygon FK the duplicate ratio of that which the side AB has to FG.

1. Let the angles AED, FLK be equal, and the sides about them proportional; and draw the diagonals EB, EC, LG, LH. Then (VI. def. 1) because the polygons are similar, the angles A and F are equal, and BA : AE :: GF : FL; wherefore, because the triangles ABE, FGL have an angle in one equal to an angle in the other, and the sides about those angles proportionals, the triangle ABE is (VI. 6) equiangular, and therefore (VI. 4 and def. 1) similar to the triangle FGL; wherefore the angles ABE, FGL are equal. Again, because the polygons are similar, the whole angle ABC is equal (VI. def. 1) to the whole FGH; therefore (I. ax. 3) the remaining angles EBC, LGH are equal:* and (VI. 4) because the triangles ABE, FGL are equiangular, EB : BA :: LG : GF; and, also (VI. def. 1) because the polygons are similar, AB : BC :: FG : GH; therefore, *ex æquo*, (V. 22) EB : BC :: LG : GH; that is, the sides about the equal angles EBC, LGH are proportionals; therefore (VI. 6) the triangles EBC, LGH are equiangular, and (VI. 4, and def. 1) similar. In the same manner, the triangles ECD, LHK may be shown to be similar. Therefore the similar polygons AD, FK are divided into the same number of similar triangles.



2. Because the triangles ABE, FGL are similar, ABE has to FGL the duplicate ratio (VI. 19) of that which the side BE has to GL. For the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL. Therefore (V. 11) as the triangle ABE to FGL, so is the triangle BEC to GLH. Again, because the triangles EBC, LGH are similar, EBC has to LGH the duplicate ratio of that which EC has to LH. For the same reason, the triangle ECD has to LHK the duplicate ratio of that which EC has to LH. As, therefore, (V. 11) the triangle EBC is to LGH, so is the triangle ECD to LHK. But it has been proved that the triangle EBC is to LGH, as the triangle ABE to FGL. Therefore (V. 11) as the triangle ABE is to FGL, so is the triangle EBC to LGH, and ECD to LHK: and therefore (V. 12) as one of the antecedents is to its consequent, so are all the antecedents to all the consequents. Wherefore as the triangle ABE is to FGL, so is the polygon AD to the polygon FK.

3. The triangle ABE has (VI. 19) to FGL, the duplicate ratio of that which the side AB has to the homologous side FG. Therefore, also, (V. 11) the polygon AD has to the polygon FK the duplicate ratio of that which AB has to the homologous side FG. Similar polygons, therefore, &c.

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* Should the angles A and F be re-entrant, so that the points A and F were on the other sides of EB and LG, the angles EBC, LGH would not be the difference, but the sum, of ABC, ABE, and of FGH, FGL.

Cor. 1. In like manner, it may be proved, that similar figures of four sides, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and the same has already been proved (VI. 19) in respect to triangles. Therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Cor. 2. If to AB, FG, two of the homologous sides a third proportional M be taken, AB has (V. def. 11) to M the duplicate ratio of that which AB has to FG. But the four-sided figure or polygon upon AB has likewise to the four-sided figure or polygon upon FG the duplicate ratio of that which AB has to FG. Therefore as AB is to M, so is the figure upon AB to the figure upon FG; which was also proved (VI. 19, cor.) in triangles. Therefore, universally, if three straight lines be proportionals, the first is to the third, as any rectilineal figure upon the first, to a similar and similarly described figure upon the second.

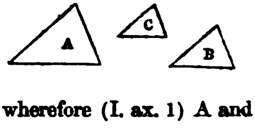
Cor. 3. Because all squares are similar figures, the ratio of any two squares to one another, is the same as the duplicate ratio of their sides; and hence, also, any two similar rectilineal figures are to one another (V. 11) as the squares described on their homologous sides.[*]

Cor. 4. The perimeters of the figures, that is, the sums of their sides, are proportional to the homologous sides. For (hyp. and VI. def. 1) the sides are proportional, and therefore, alternately, AB : FG :: BC : GH :: CD : HK :: DE : KL :: EA : LF: wherefore (V. 12) AB : FG :: AB+BC+CD+DE+EA : FG+GH+HK+KL+LF.

PROP. XXI. THEOR.—Rectilineal figures which are similar to the same figure, are similar to one another.

Let each of the rectilineal figures A, B be similar to C: A is similar to B.

Because A is similar to C, they are (VI. def. 1) equiangular, and have also their sides about the equal angles proportional. For the same reason, B and C are equiangular, and have their sides about the equal angles proportional. Therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them, and of C proportional; wherefore (I. ax. 1) A and



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[*] This corollary and the preceding, afford easy means of comparing the magnitudes of similar rectilineal figures. Thus, the second enables us to find two lines having the same ratio as the figures, and the third affords, in general, the easiest means of exhibiting their ratio in numbers, or by algebraic symbols; the area of a square (I. 46, cor. 5) being found simply by multiplying a side by itself. Thus, if AB were 3 inches, and FG 2 inches, the areas of the squares of these lines would be 9 square inches and 4 square inches; and therefore the figure AD would be to FK as 9 to 4, or would be rather more than twice as great. If, again, AB were 5 feet, and FG 3 feet, AD would be to FK as 25 to 9, or would be nearly three times as great.

B are equiangular; and (V. 11) they have their sides about the equal angles proportionals. Therefore (VI. def. 1) A is similar to B. Rectilineal figures, therefore, &c.

Prop. XXII. Theor.—If four straight lines be proportionals, the similar rectilineal figures, similarly described upon them, are also proportionals: and (2) if the similar rectilineal figures, similarly described upon four straight lines, be proportionals, those lines are proportionals.

Let AB : CD :: EF : GH; and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and upon EF, GH the similar rectilineal figures MF, NH in like manner: KAB is to LCD, as MF to NH.

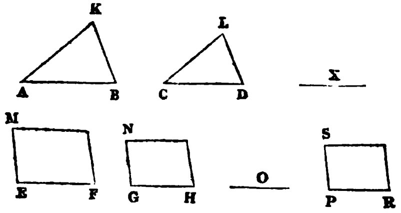
To AB, CD take (VI. 11) a third proportional X; and to EF, GH a third proportional O. Then, because

AB : CD :: EF : GH, and (V. 11) CD : X :: GH : O;

therefore, *ex æquo*, as AB : X :: EF : O. But (VI. 20, cor. 2) as AB to X, so is the figure KAB to LCD; and as EF to O, so is MF to NH: therefore (V. 11) as KAB to LCD, so is MF to NH.

Again, if the rectilineal figure KAB be to LCD, as MF to NH; AB : CD :: EF : GH.

To AB, CD, and EF, take (VI. 12) a fourth proportional PR, and on it describe (VI. 18) the figure SR similar and similarly situated to either of the figures MF, NH. Then, because AB : CD :: EF : PR, and that upon AB, CD are described the similar and similarly situated rectilineal figures KAB, LCD, and upon EF, PR, in like manner, the similar rectilineal figures MF, SR; KAB : LCD :: MF : SR; but (hyp.) KAB : LCD :: MF : NH; and therefore MF having (V. 11) the same ratio to each of the two NH, SR, these are equal (V. 9) to one another. They are also similar and similarly situated; therefore GH is equal to PR: and because as AB : CD :: EF : PR, and that PR is equal to GH; therefore AB : CD :: EF : GH; wherefore, &c.

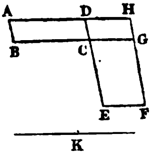


Prop. XXIII. Theor.—Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angles

BCD, ECG equal: AC has to CF the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC, CE are also (I. 14, cor.) in a straight line. Complete the parallelogram DG. To DC, CE, CG take (VI. 12) a fourth proportional K. Then (V. def. 10) the ratio of BC to K is said to be compounded of the ratios of BC to CG and of CG to K, or (const.) of the ratios of BC to CG and DC to CE: wherefore BC has to K the ratio compounded of the ratios of the sides. Now (VI. 1) as BC to CG, so is AC to CH; and as DC to CE, so is CH to CF, or (const.) as CG to K, so is CH to CF. Therefore, since there are two ranks of magnitudes,



BC, CG, K, and AC, CH, CF,

which, taken two and two in order, are proportionals, *ex æquo* BC is to K, as AC to CF. But it has been shown that BC has to K the ratio compounded of the ratios of the sides; and therefore AC has to CF the ratio compounded of the ratios of the sides; that is, of the ratios of BC to CG, and DC to CE. Therefore, equiangular parallelograms, &c.*

Cor. Triangles which have one angle of the one equal, or supplemental, to one angle of the other, have to one another the ratio which is compounded of the ratios of the sides containing those angles. For, if BD and EG be joined, the triangles BCD, ECG being (I. 34) halves of the parallelograms AC, CF, have (V. 15) the same ratio as the parallelograms; and if BD and CF be joined, the same is true regarding the triangles BCD, CEF, which have the angles BCD, CEF supplemental.

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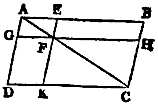
* Otherwise:—Because there are three parallelograms AC, CH, CF, the first AC (V. def. 10) has to the third CF, the ratio which is compounded of the ratio of the first AC to the second CH, and of the ratio of CH to the third CF; but AC is to CH, as the straight line BC to CG; and CH is to CF, as the straight line CD is to CE; therefore AC has to CF the ratio which is compounded of ratios that are the same with the ratios of the sides.

The demonstration in the text proceeds on the same principle as that given by Dr. Simson from the Greek of Euclid, but it is considerably simplified by using the lines BC, CG, instead of two lines proportional to them. The foregoing proof, in this Note, is given by Simson in his Notes, from Candalla. As a proof, the latter is quite satisfactory; but the other has the advantage of exhibiting in a clear manner the ratio which is compounded of the ratios of the sides, as it is simply that of BC to K.

Dr. Simson remarks in his Note on this proposition, that "nothing is usually reckoned more difficult in the elements of geometry by learners, than the doctrine of compound ratio." This distinguished geometer, however, has both freed the text of Euclid from the errors introduced by Theon or others, and has explained the subject in such a manner as to remove the difficulties that were formerly felt. According to him, "every proposition in which compound ratio is made use of, may without it be both enunciated and demonstrated;" and "the use of compound ratio consists wholly in this, that by means of it, circumlocutions may be avoided, and thereby propositions may be more briefly either enunciated or demonstrated, or both may be done. For instance, if this 23d proposition of the sixth book were to be enunciated, without mentioning compound ratio, it might be done as follows:— If two parallelograms be equiangular, and if as a side of the first to a side of the second, so any assumed straight line be made to a second straight line; and as the

Prop. XXIV. Theor.—The parallelograms about the diagonal of any parallelogram are similar to the whole, and to one another.

Let ABCD be a parallelogram, and EG, HK the parallelograms about the diagonal AC: the parallelograms EG, HK are similar to the whole parallelogram, and to one another.



Because DC, GF are parallels, the angles ADC, AGF are (I. 29) equal. For the same reason, because BC, EF are parallels, the angles ABC, AEF are equal: and (I. 34) each of the angles BCD, EFG is equal to the opposite angle DAB, and therefore they are equal to one another: wherefore in the parallelograms the angle ABC is equal to AEF, and BAC common to the two triangles BAC, EAF; therefore (VI. 4, cor.) as AB : BC :: AE : EF. And, (V. 7) because the opposite sides of parallelograms are equal to one another, AB : AD :: AE : AG; and DC : BC :: GF : EF; and also CD : DA :: FG : GA. Therefore the sides of the parallelograms BD, EG about the equal angles are proportionals: the parallelograms are, therefore, (VI. def. 1) similar to one another. In

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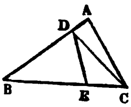
other side of the first to the other side of the second, so the second straight line be made to a third: the first parallelogram is to the second, as the first straight line to the third; and the demonstration would be exactly the same as we now have it. But the ancient geometers, when they observed this enunciation could be made shorter, by giving a name to the ratio which the first straight line has to the last, by which name the intermediate ratios might likewise be signified, of the first to the second, and of the second to the third, and so on, if there were more of them, they called this ratio of the first to the last, the ratio compounded of the ratios of the first to the second, and of the second to the third straight line: that is, in the present example, of the ratios which are the same with the ratios of the sides."

The proposition in the text will be illustrated by the following one, which exhibits the subject in a different, and, in some respects, a preferable light.

Triangles which have an angle of the one equal to an angle of the other, are proportional to the rectangles contained by the sides about those angles: and (2) *equiangular parallelograms are proportional to the rectangles contained by their adjacent sides.*

1. Let ABC, DBE be two triangles, having the angles ABC, DBE equal: the first triangle is to the second as AB.BC is to DB.BE.

Let the triangles be placed with their equal angles coinciding, and join CD. Then (VI. 1) AB is to DB, as the triangle ABC to DBC. But (VI. 1, cor. 2) AB : DB :: AB.BC : DB.BC; therefore (V. 11) AB.BC is to DB.BC, as the triangle ABC to DBC. In the same manner, it would be shown that DB.BC is to DB.BE, as the triangle DBC to DBE; and, therefore, *ex æquo*, AB.BC is to DB.BE, as the triangle ABC to DBE.



2. If parallels to BC through A and D, and to AB through C and E were drawn, parallelograms would be formed which would be respectively double of the triangles ABC and DBE, and which (V. 15) would have the same ratio as the triangles; that is, the ratio of AB.BC to DB.BE: and this proves the second part of the proposition.

Comparing this proposition and the 23d, we see that *the ratio which is compounded of the ratios of the sides, is the same as the ratio of their rectangles, or the same* (I. 46, cor. 5) *as the ratio of their products, if they be expressed in numbers.* This conclusion might also be derived from the proof given in the text. For (const.) DC : CE :: CG : K; whence (VI. 16) K.DC = CE.CG. But, it was proved that BC : K :: AC : CF; or (VI. 1) BC.DC : K.DC :: AC : CF; or BC.DC : CE.CG :: AC : CF, because K.DC = CE.CG.

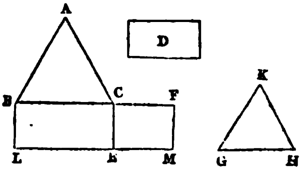
The 14th proposition of this book is evidently a case of this proposition; and the 19th is also easily derived from it.

the same manner, it would be shown that the parallelogram BD is similar to HK. Therefore each of the parallelograms EG, HK is similar to BD. But (VI. 21) rectilineal figures which are similar to the same figure, are similar to one another; therefore the parallelogram EG is similar to HK.* Wherefore, &c.

PROP. XXV. PROB.—To describe a rectilineal figure which shall be similar to one given rectilineal figure, and equal to one another.

Let ABC and D be given rectilineal figures. It is required to describe a figure similar to ABC, and equal to D.

Upon the straight line BC describe (I. 45, cor.) the parallelogram BE equal to ABC; also upon CE describe the parallelogram CM equal to D, and having the angle FCE equal to CBL. Therefore (I. 29, and 14) BC and CF are in a straight line, as also LE and EM. Between BC and CF find (VI. 13) a mean proportional GH, and on it describe (VI. 18) the figure GHK similar, and similarly situated, to ABC; GHK is the figure required.



Because BC : GH :: GH : CF, and if three straight lines be proportionals, as the first is to the third, so is (VI. 20, cor. 2) the figure upon the first to the similar and similarly described figure upon the second; therefore,

as BC to CF, so is ABC to KGH; but (VI. 1)
as BC to CF, so is BE to EF:

therefore (V. 11) as ABC is to KGH, so is BE to EF. But (const.) ABC is equal to BE; therefore KGH is equal (V. 14) to EF; and (const.) EF is equal to D; wherefore also KGH is equal to D; and it is similar to ABC. Therefore the rectilineal figure KGH has been described, similar to ABC, and equal to D: which was to be done.

PROP. XXVI. THEOR.—If two similar parallelograms have a common angle, and be similarly situated; they are about the same diagonal.

Let the parallelograms BD, EG be similar and similarly situated, and have the angle DAB common: they are about the same diagonal.

For, if not, let, if possible, the parallelogram BD have its diagonal AHC in a different straight line from AF, the diagonal of

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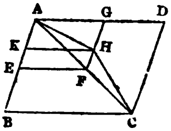
* Hence, GF : FE :: FH : FK. Therefore the sides of the parallelograms GK and EH, about the equal angles at F, are reciprocally proportional; and (VI. 14) these parallelograms are equal; a conclusion which agrees with the 43d proposition of the first book.

EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC. Therefore, (VI. 24) since the parallelograms BD, KG are about the same diagonal, they are similar to one another; wherefore, (VI. def. 1)

DA : AB :: GA : AK.



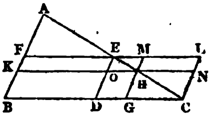
But (VI. def. 1) because BD, and EG are similar,

DA : AB :: GA : AE;

therefore (V. 11) as GA : AE :: GA : AK; wherefore (V. 9) AK is equal to AE, the less to the greater, which is impossible. Therefore AHC is not the diagonal of BD; and in the same manner it may be shown that no other line, except AFC, is the diagonal of BD: AFC therefore is its diagonal, and BD, EG are about the same diagonal AFC. Therefore, &c.

PROP. XXVII. THEOR.—Of all the parallelograms that can be inscribed in any triangle, that which is described on the half of one of the sides as base, is the greatest.

Let ABC be a triangle, having BC, one of its sides, bisected in D; draw (I. 31) DE parallel to BA, and EF to BC: let also G be any other point in BC, and describe the parallelogram GK; FD is greater than KG.



If G be in DC, through C draw CL parallel to BA, and produce FE, KH, GH, as in the figure. Then (I. 43) the complements LH and HD are equal; and since the bases CD, DB are equal, the parallelograms ND, DK (I. 36) are equal. To LH add ND, and to HD add DK: then (I. ax. 2) the gnomon MND is equal to the parallelogram KG. But (I. ax. 9) DL is greater than MND; and therefore FD, which (I. 36) is equal to DL, is greater than KG, which is equal to the gnomon MND.

If G were in BD, since BD is equal to DC, AE is equal (VI. 2) to EC, and AF to FB; and by drawing through A a parallel to BC, meeting DE produced, it would be proved in the same manner, that FD is greater than the inscribed parallelogram applied to BG. Therefore, of all the parallelograms, &c.*

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* The parallelogram FD exceeds KG by the parallelogram OM similar to DL or DF, and described on OH, which is equal to DG, the difference of the bases BD and BG. Hence we derive the solution of the next proposition.

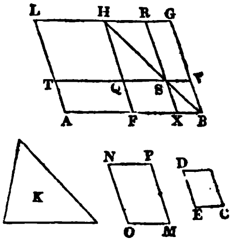
The enunciation of this proposition here given is much more simple and intelligible than that of Euclid, and the proof is considerably shortened. Euclid's enunciation, as given by Dr. Simson, is as follows:—"Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest." It may be remarked, that this proposition, in its simplest case, is the same as the second corollary to the fifth proposition of the second book.

Respecting this proposition and the next, see the Notes at the end of the book.

Cor. Since (VI. 24) all parallelograms having one angle coinciding with BCL, and their diagonals with CA, are similar, it follows from this proposition, that if, on the segments of a given straight line BC, two parallelograms of the same altitude be described, one of them DL, similar to a given parallelogram, the other DF will be the greatest possible, when the segments of the line are equal.

PROP. XXVIII. PROB.—To divide a given straight line into two parts, such that parallelograms of equal altitudes may be described upon them, one equal to a given rectilineal figure, and the other similar to a given parallelogram; the rectilineal figure not being greater (VI. 27) than the parallelogram described on half the given line, and similar to the given parallelogram.

Let AB be a given straight line, and CD a given parallelogram, of which EC is a side; let AB be bisected (I. 10) in F; and on FB (VI. 18) let the parallelogram FG be described, similar to CD, and having FB homologous to EC; let also K be a rectilineal figure, which is not greater than FG: it is required to divide AB into two parts, such, that if on one of them a parallelogram be described, similar to CD, and having that part homologous to EC, the parallelogram on the other, of equal altitude, may be equal to K.



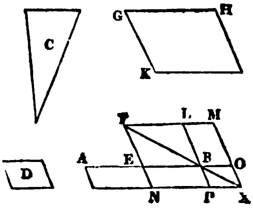
Through A draw (I. 31) AL parallel to FH, and produce GH to meet it in L. Then, if AH be equal to K, the thing is done: for on FB one of the parts of AB, the parallelogram FG is described similar to CD, and similarly situated; and on AF, the remaining part, a parallelogram AH, of equal altitude, is described, equal to the given rectilineal figure K.

But if AH be not equal to K, it is greater by the determination: therefore also FG, which (I. 36) is equal to AH, is greater than K. Make (VI. 25) the parallelogram MN similar to CD or FG, and equal to the excess of FG above K. Let the side NO be homologous to HF or DE, and consequently NP to HG; and since FG is equal to K and MN taken together, it is plain that HF, HG are respectively greater than NO, NP. Cut off, therefore, HQ, HR respectively equal to NO, NP, and complete the parallelogram HQSR; produce also the sides QS and RS. Then, since OP is similar to FG, QR, which is equal and similar to OP, is also (VI. 21) similar to FG, and therefore (VI. 26) QR and FG are about the same diagonal. Let HSB be their common diagonal. Now, since (const.) K and OP are together equal to FG, take away OP and QR, which are equal, and (I. ax. 3) there remains the figure K equal to the gnomon RVF. But it may be shown as in the last proposition, that TX is equal to the gnomon RVF; therefore TX is equal to K; and AB is divided into two parts AX, XB, such that on one of them the parallelogram TX is described equal to K, and on the other the parallelogram XV, of the same altitude, is described similar to the given parallelogram CD, and having the segment XB, of the given line, homologous to EC; which was to be done.

Prop. XXIX. Prob.—To produce a given straight line so that a parallelogram similar to a given one being described on the produced part, another parallelogram of equal altitude described on the whole line produced, may be equal to a given rectilineal figure.

Let AB be a given straight line, C a given rectilineal figure, and D a given parallelogram: it is required to produce AB, so that if a parallelogram be described on the produced part similar to D, and similarly situated, a parallelogram of the same altitude on the whole produced line shall be equal to C.

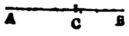
Bisect AB in E, and (VI. 18) on EB describe the parallelogram EL similar, and similarly situated to D: and make (VI. 25) the parallelogram KH equal to EL and C together, and similar, and similarly situated to D; wherefore (VI. 21, and const.) KH is similar to EL. Let GH be the side homologous to FL, and GK to FE: and because KH is greater than EL, the side GH is greater than FL, and GK than FE. Produce FL and FE, and make FLM equal to GH, and FEN to GK, and complete the parallelogram MN. MN is therefore equal and similar to KH; but KH is similar to EL; wherefore (VI. 21) MN is similar to EL, and consequently (VI. 26) EL and MN are about the same diagonal: draw their diagonal FBX, and complete the figure. Therefore, since KH is equal to EL and C together, and that KH is equal to MN; MN is equal to EL and C: take away the common part EL; then the remaining gnomon NOL is equal to C. And because AE is equal to EB, the parallelogram AN is equal (I. 36) to NB, that is (I. 43) to BM. Add NO to each; therefore the whole parallelogram AX, is equal to the gnomon NOL. But NOL is equal to C; therefore also AX is equal to C. Wherefore, to the whole produced line AO there is applied the parallelogram AX equal to the given figure C; and the parallelogram PO, of the same altitude as AX, is described on the produced part BO, and is similar to D, because (VI. 24) PO is similar to EL: which was to be done.



PROP. XXX. PROB.—To divide a given straight line in extreme and mean ratio.

Let AB be the given straight line; it is required to divide it in extreme and mean ratio.

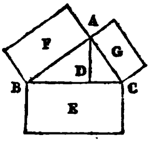
Divide AB (II. 11) in the point C, so that the rectangle AB.BC may be equal to the square of AC: then (VI. 17) because the rectangle AB.BC is equal to the square of AC, as BA : AC :: AC : CB: therefore (VI. def. 3) AB is cut in extreme and mean ratio in C: which was to be done.



PROP. XXXI. THEOR.—The rectilineal figure described upon the hypotenuse of a right-angled triangle, is equal to the similar and similarly described figures upon its other sides.

Let ABC be a right-angled triangle, having the right angle BAC: the rectilineal figure E, described upon BC, is equal to F, G, the similar figures, similarly described upon BA, AC.

Draw the perpendicular AD. Then, because AD is drawn from the right angle perpendicular to the hypotenuse, the triangles ABD, ADC are similar (VI. 8) to the whole triangle ABC, and to one another; and therefore (VI. 4) as CB : BA :: BA : BD; and (VI. 20, cor. 2) because these three straight lines are proportionals, as CB : BD :: E : F: and inversely, (V. B) as DB : BC :: F : E. For the same reason, as DC : BC :: G : E. Wherefore (V. 24) as BD and DC together are to BC, so are the figures F and G together to the figure E. But BD and DC together are equal to BC: therefore (V. A) the figure E is equal to the figures F and G. Wherefore, &c.

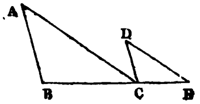


Schol. The 47th proposition of the first book is the case of this proposition, in which the figures described on the three sides are squares.

PROP. XXXII. THEOR.—If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another, and the angles contained by those sides, opening towards the same parts; the remaining sides are in a straight line.

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE, viz. BA : AC :: CD : DE; and let AB be parallel to DC, and AC to DE, the angles A and D opening in the same direction; BC, CE are in a straight line.

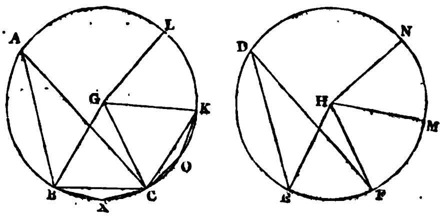
Because AB is parallel to DC, and AC meets them, the alternate angles BAC, ACD are (I. 29) equal. For the same reason, the angle CDE is equal to ACD: wherefore also (I. ax. 1) BAC is equal to CDE. And because the triangles ABC, DCE have the angles A and D equal, and the sides about these angles proportionals, viz. BA : AC :: CD : DE, these triangles are (VI. 6) equiangular. Therefore the angles ABC, DCE are equal: and the angle BAC was proved to be equal to ACD. Therefore the whole ACE is equal to the two ABC, BAC; add ACB; then the angles ACE, ACB are equal to the angles ABC, BAC, ACB; that is (I. 32) to two right angles. Now, since at the point C in AC, the two lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore (I. 14) BC and CE are in a straight line. If, therefore, &c.



Prop. XXXIII. Theor.—In equal circles, or in the same circle, angles, whether at the centres or circumferences, have the same ratio, as the arcs on which they stand, have to one another: and so also have the sectors.

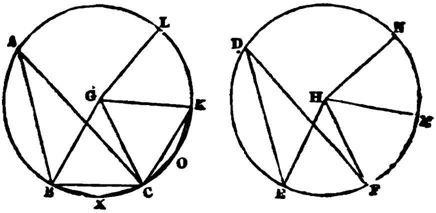
Let ABC, DEF be equal circles; and let BGC, EHF be angles at their centres, and BAC, EDF be angles at their circumferences; as the arc BC to the arc EF, so is the angle BGC to EHF, and the angle BAC to EDF; and also the sector BGC to the sector EHF.

Take any number of arcs CK, KL, each equal to BC, and any number FM, MN, each equal to EF: and join GK, GL, HM, HN. Because the arcs BC, CK, KL are all equal, the angles BGC, CGK, KGL are also (III. 27) all equal: therefore whatever multiple the arc BL is of BC, the same multiple is the angle BGL of BGC. For the same reason, whatever multiple the arc EN is of EF, the same multiple is the angle EHN of EHF. Also, if the arc BL be equal to EN, the angle BGL is also equal



(III. 27) to EHN; and if BL be greater than EN, likewise the angle BGL is greater than EHN; and if less, less. Therefore (V. def. 5) the arc BC is to the arc EF, as the angle BGC to EHF. But (V. 15) the angle BGC is to EHF, as the angle BAC to EDF, for (III. 20) each is double of each: therefore (V. 11) as the arc BC is to EF, so is the angle BGC to EHF, and the angle BAC to EDF.

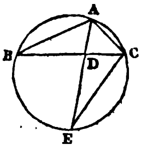
Also, as the arc BC is to EF, so is the sector BGC to the sector EHF. In the arcs BC, CK, take any points X, O, and join BC, CK, BX, XC, CO, OK. Then, because in the triangles GBC, GCK, the sides BG, GC are equal to CG, GK, and that they contain equal angles, BC is equal (I. 4) to CK, and the triangle GBC to GCK. Again, because the arc BC is equal to the arc CK, the remaining part BAC of the whole circumference, is equal to the remaining part CAK: wherefore (III. 27) the angles BXC, COK are equal; and the segment BXC is therefore similar (III. def. 8) to the segment COK; and they are upon equal straight lines BC, CK: they are therefore equal (III. 24) to one another. And since the triangles BGC, CGK are equal, therefore (I. ax. 2) the whole, the sector BGC, is equal to the whole, the sector CGK. For the same reason, the sector KGL is equal to each of the sectors BGC, CGK. In like manner, the sectors EHF, FHM, MHN may be proved equal to one another. Therefore, whatever multiple the arc BL is of BC, the same multiple is the sector BGL of BGC; and whatever multiple the arc EN is of EF, the same multiple is the sector EHN of EHF: and if the arc BL be equal to EN, the sector BGL is equal to EHN; if the arc BL be greater than EN, the sector BGL is greater than EHN; and if less, less. Therefore (V. def. 5) as the arc BC is to EF, so is the sector BGC to EHF. Wherefore, in equal circles, &c.



PROP. B. THEOR.—If an angle of a triangle be bisected by a straight line which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by AD: the rectangle BA.AC is equal to the rectangle BD.DC, together with the square of AD.

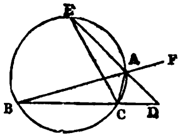
Describe the circle (IV. 5) ACB about the triangle; produce AD to meet the circumference in E: and join EC. Then (hyp.) the angles BAD, CAE are equal; as are also (III. 21) the angles B and E, for they are in the same segment: therefore (VI. 4, cor.) in the triangles ABD, AEC, as BA : AD :: EA : AC; and consequently (VI. 16) the rectangle BA.AC is equal to EA.AD, that is (II. 3) ED.DA, together with the square of AD. But (III. 35) the rectangle ED.DA is equal to the rectangle BD.DC: therefore the rectangle BA.AC is equal to BD.DC, together with the square of AD:* wherefore, if an angle, &c.



Prop. C. Theor.—If an exterior angle of a triangle be bisected by a straight line, which cuts the base produced; the rectangle contained by the sides of the triangle, and the square of the bisecting line, are together equal to the rectangle contained by the segments of the base intercepted between its extremities and the bisecting line.

Let the exterior angle CAF of the triangle BAC be bisected by AD: the rectangle BA.AC, and the square of AD are together equal to the rectangle BD.DC.

Describe (IV. 5) the circle AEBC about the triangle ABC: produce DA to E; and join EC. Then, since (hyp.) the angles CAD, FAD are equal, their supplements CAE, DAB (I. def. 38, and ax. 3) are also equal; and (III. 21) B and E are equal. Therefore, (VI. 4, cor.) in the triangles BAD, EAC, as BA : AD :: EA : AC; and consequently (VI. 16) the rectangle BA.AC is equal to EA.AD. To each add the square of AD; therefore $BA.AC + AD^2 = EA.AD + AD^2$, or (II. 3) $BA.AC + AD^2 = ED.DA$; or (III. 36, cor.) $BA.AC + AD^2 = BD.DC$. Therefore, if an exterior angle, &c.



Prop. D. Theor.—If from an angle of a triangle a perpendicular be drawn to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

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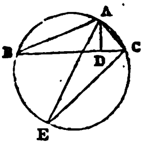
* From this proposition in connexion with the 3d proposition of this book, we have the means of computing AD, when the sides are given in numbers. For, by the 3d, BA : AC :: BD : DC; and, by composition, BA+AC : AC :: BC : DC. This analogy gives DC, and BD is then found by taking DC from BC. But by this proposition $BA.AC = BD.DC + AD^2$; therefore, from BA.AC, take BD.DC, and the square root of the remainder will be AD.

Let ABC be a triangle, AD the perpendicular from the angle A to BC; and AE a diameter of the circumscribed circle ABEC: the rectangle BA.AC is equal to the rectangle AD.AE.

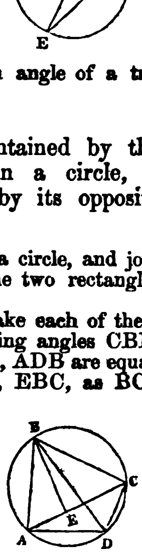


Join EC. Then the right angle BDA is equal (III. 31) to the angle ECA in a semicircle, and (III. 21) the angle B to the angle E in the same segment; therefore (VI. 4, cor.) as BA : AD :: EA : AC; and consequently (VI. 16) the rectangle BA.AC is equal to the rectangle EA.AD. If, therefore, from an angle of a triangle, &c.

Prop. E. Theor.—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let ABCD be a quadrilateral inscribed in a circle, and join AC, BD; the rectangle AC.BD is equal to the two rectangles AB.CD, and AD.BC.

Make the angle ABE equal to DBC, and take each of them from the whole angle ABC; then the remaining angles CBE, ABD are equal; and (III. 21) the angles ECB, ADB are equal. Therefore (VI. 4, cor.) in the triangles ABD, EBC, as BC : CE :: BD : DA; whence (VI. 16) BC.DA= BD.CE. Again, in the triangles BAE, BDC, because (const.) the angles ABE, DBC are equal, as also (III. 21) BAE, BDC, therefore (VI. 4, cor.) as BA : AE :: BD : DC; whence (VI. 16) BA.DC=BD.AE. Add these equal rectangles to the equals BC.DA and BD.CE: then BA.DC+BC.DA=BD.CE+BD.AE, or (II. 1) BA.DC+BC.DA= BD.AC. Therefore the rectangle, &c.



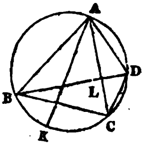
Cor. 1. If the sides AD, DC, and consequently (III. 28, and 27) the arcs AD, DC, and the angles ABD, CBD be equal, the rectangle BD.AC is equal to AB.AD together with BC.AD, or (II. 1) to the rectangle under AD, and the sum of AB and BC. Hence (VI. 16) AB+BC : BD :: AC : AD or DC.

Cor. 2. If AC, AD, CD be all equal, the last analogy becomes AB+BC : BD :: AD : AD; whence AB+BC=BD. Hence in an equilateral triangle inscribed in a circle, a straight line drawn from the vertex to a point in the arc cut off by the base is equal to the sum of the chords drawn from that point to the extremities of the base.

In a similar manner, from propositions A and C, the line bisecting the exterior vertical angle may be computed: and from proposition D, in connexion with the 12th or 13th of the second book, the diameter of the circumscribed circle may be computed, when the sides of the triangle are given in numbers.

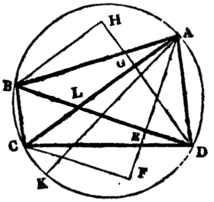
Prop. F. Theor.—The diagonals of a quadrilateral inscribed in a circle, are proportional to the sums of the rectangles contained by the sides meeting at their extremities.

Let ABCD be a quadrilateral inscribed in a circle, and AC, BD its diagonals; AC : BD :: BA.AD+BC.CD : AB.BC+AD.DC.



If AC, BD cut one another perpendicularly in L, then, (VI. D) AK being the diameter of the circle, AL.AK=BA.AD, and CL.AK=BC.CD; whence, by addition (I. ax. 2) AL.AK+CL.AK or (II. 1) AC.AK=BA.AD+BC.CD. In a similar manner, it would be proved, that BD.AK=AB.BC+AD.DC. Hence AC.AK : BD.AK, or (VI. 1) AC : BD :: BA.AD+BC.CD : AB.BC+AD.DC.

But if AC be not perpendicular to BD, draw AEF perpendicular and CF parallel to BD, and DGH perpendicular and BH parallel to AC. Then, because EF is equal to the perpendicular drawn from C to BD, and GH equal to the one drawn from B to AC; it would be proved as before, that AF.AK =BA.AD+BC.CD, and DH.AK= AB.BC+AD.DC. Hence, AF.AK : DH.AK, or (VI. 1) AF : DH :: BA.AD +BC.CD : AB.BC+AD.DC. But the triangles AFC, DHB are equiangular, having the right angles F and H, and the angles ACF, DBH, each equal (I. 29) to ALD; therefore (VI. 4) AF : AC :: DH : DB, and alternately AF : DH :: AC : DB. Hence (V. 11) the foregoing analogy becomes AC : BD :: BA.AD+BC.CD : AB.BC+AD.DC. Wherefore, the diagonals, &c.*



Prop. G. Theor.—If in a straight line drawn through the centre of a circle, and on the same side of the centre, two points be taken so that the radius is a mean proportional between their distances from the centre; two straight lines drawn from those points to any point whatever in the

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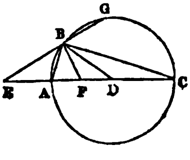
* From this proposition and the last, when the sides of a quadrilateral inscribed in a circle are given, we can find the ratio of the diagonals and their rectangle, and thence (VI. 25) the diagonals themselves. Also, if the sides be given in numbers, we can compute the diagonals. Thus, let the sides taken in succession round the figure be 50, 78, 104, and 120. Then, the ratio of the diagonals will be that of 50×78+104×120 to 50×120+78×104; that is, 16380 to 14112, or 65 to 56, by dividing by 252. Again, the rectangle of the diagonals is 50×104+78×120, or 14560. But (VI. 20, cor. 3) similar rectilineal figures are as the squares of the corresponding sides, and consequently the sides are as the square roots of the areas: therefore, taking 65 and 56 as the sides of a rectangle, we have its area equal to 3640; and $\sqrt{3640}$ is to $\sqrt{14560}$, or $\sqrt{3640}$ is to $\sqrt{(4\times3640)}$, that is, 1 : 2 :: 65 : 130 :: 56 : 112; so that 130 and 112 are the diagonals.

circumference, are proportional to the segments into which the circumference divides the straight line intercepted between the same points.

Let ABC be a circle, and CAE a straight line drawn through its centre D; if ED : DA :: DA : DF, and if BE, BF be drawn from any point B of the circumference; EB : BF :: EA : AF.

Join AB, BD. Then, since DB is equal to DA, we have (hyp.) ED : DB :: DB : DF. The two triangles EDB, BDF, therefore, have their sides about the common angle D proportional; wherefore (VI. 6) the angle E is equal to FBD. Now the angle BAD is equal (I. 32) to the two angles E and EBA, and also (I. 5) to ABD: wherefore E and EBA are (I. ax. 1) equal to ABD. From these take the equal angles E and FBD, and (I. ax. 3) the remaining angles EBA, ABF are equal: and therefore, (VI. 3) in the triangle EBF, EB : BF :: EA : AF. If, therefore, in a straight line, &c.



Cor. Join BC, and produce EB to G. Then, since ABC is (III. 31) a right angle, it is equal to the two EBA, CBG. From these equals take the equal angles ABF, EBA, and the remainders FBC, CBG are equal: and therefore (VI. A) EB : BF :: EC : CF. But it has been proved, that EB : BF :: EA : AF; therefore (V. 11) EA : AF :: EC : CF. Hence, if the segments EA, AF be given, the point C may be determined by the method shown in the scholium to the 10th proposition of this book; and the circle ABC may then be described, its diameter AC being determined.[*]

PROP. H. THEOR.—**The perpendiculars drawn from the three angles of any triangle to the opposite sides, intersect one another in the same point.**

If the triangle be right-angled, it is plain that all the perpendiculars pass through the right angle. But if it be not right-angled, let ABC be the triangle, and about it describe a circle: then, B and C being acute angles,[†] draw ADE perpendicular to BC, cut-

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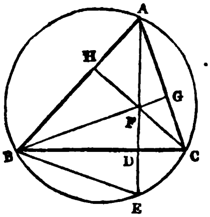
[*] The circle may also be determined in the following manner. Since (hyp.) ED : DA :: DA : DF, by division, EA : DA :: AF : DF; whence, alternately and by division, EA — AF : AF :: AF : DF. Hence DF is a third proportional to the difference of EA, AF, and to AF, the less; and thus the centre D is determined. From the last analogy also we obtained (V. E. cor.) EA — AF : EA :: AF : AD; an analogy which serves the same purpose, since it shows that the radius of the circle is a fourth proportional to the difference of the segments EA, AF, and to those segments themselves.

[†] This limitation prevents the necessity of a different case, which would arise if the perpendicular AD fell without the triangle. If the angle A be obtuse, the point F lies without the circle, and BF, not produced, cuts AC produced. The proof, however, is the same. The demonstration here given is new, so far as is known to the present editor; and it is very easy and obvious. Another easy and elegant

ting BC in D, and the circumference in E; and make DF equal to DE: join BF and produce it, if necessary, to cut AC, or AC produced in G: BG is perpendicular to AC. Join BE: and because FD is equal to DE, the angles at D right angles, and DB common to the two triangles FDB, EDB, the angle FBD is equal (I. 4) to EBD: but (III. 21) CAD, EBD are also equal, because they are in the same segment; therefore CAD is equal to FBD or GBC. But the angle ACB is common to the two triangles ACD, BCG; and therefore (I. 32. cor. 5) the remaining angles ADC, BGC are equal: but (const.) ADC is a right angle; therefore also BGC is a right angle, and BG is perpendicular to AC. In the same manner it would be shown, that a straight line CH, drawn through C and F, is perpendicular to AB. The three perpendiculars therefore all pass through F: wherefore, the perpendiculars, &c.



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proof, of which the following is an outline, is given in Garnier's "Réciproques," &c. theor. III. page 78: Draw BG and CH perpendicular to AC and AB; join GH, and about the quadrilaterals AHFG and BHGC describe circles, which can be done, as is easily shown: draw also AFD. Then the angles BAD, BCH are equal, each of them being equal (III. 21) to HGF; and the angle ABC being common, ADB is equal (I. 32) to BHC, and is therefore a right angle.

ELEMENTS OF PLANE TRIGONOMETRY.[*]

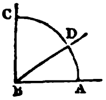
DEFINITIONS.

1. Two perpendicular diameters divide a circle into four parts called *quadrants*. These are evidently all equal to one another.[†]

2. If a right angle be divided into 90 equal parts, each of the parts is called a *degree*. If a degree be divided into 60 equal parts, each is called a *minute*; and if a minute be divided into 60 equal parts, each of them is called a *second*.[‡]

Degrees, minutes, seconds are usually denoted by the marks $^\circ$, $'$, $''$. Thus, $23^\circ\ 27'\ 33''$ is read 23 degrees, 27 minutes, 33 seconds.

3. If from the vertex of a right angle as centre, a circle be described, all equal angles at the centre will stand (Euc. III. 26) on equal arcs of the circumference. The 90 equal angles of one degree each will therefore stand on 90 equal arcs. The several angles of one minute each will also stand on equal arcs; and so likewise will all the angles of one second each. The divisions thus obtained are called degrees, minutes, and seconds of the circumference; and it is plain, that as many degrees, minutes, and seconds as any angle contains of the one kind, so many of the other kind will the arc on which it stands contain.



Thus, if ABC be a right angle, the arc AC will contain 90° degrees of the circumference; and if ABD be an angle of $38^\circ\ 25'$, the arc AD will consist of 38 degrees, 25 minutes of the circumference.

It is plain, conversely (Euc. III. 27), that as many degrees, &c. as are in any circular arc, so many degrees, &c. are there in the angle at the centre, which stands on that arc. On this account a circular arc is often called the *measure* of the angle which stands on it at the centre.

Hence, the arc of a semicircle contains 180 degrees, and the entire circumference 360 degrees.

4. The difference between an angle and a right angle, or between

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[*] Plane Trigonometry, in its primitive meaning, is that branch of mathematical science which determines by computation, certain sides or angles of a plane triangle, by means of others that are given in numbers. In its extended and improved state, it is a powerful instrument of investigation in many of the higher speculations in mathematics. The short tract here given will be confined to the former object.

[†] As the circumference of a circle is sometimes called the circle, so the *arc of a quadrant* is often called a quadrant.

[‡] In several modern French works, the right angle is divided into 100 equal parts called degrees, each degree into 100 equal parts called minutes, and each minute into 100 equal parts called seconds.

an arc and a quadrant, is called the *complement* of that angle or arc.

5. The straight line drawn from one extremity of an arc, perpendicular to the diameter passing through the other extremity, is called the *sine* of that arc, or of the angle measured by it: and the part of that diameter intercepted between the sine and the arc, is called the *versed sine* of the arc, or of the angle which it measures.

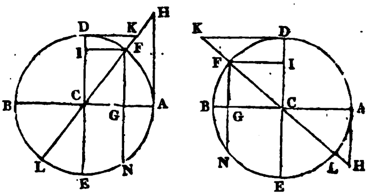
6. If a straight line touch a circle at one extremity of an arc, the part of it intercepted between that extremity and the diameter produced, which passes through the other, is called the *tangent* of the arc, or of the angle which it measures; and the straight line drawn from the centre to the remote extremity of the tangent, is called the *secant* of the arc or angle.

7. The *cosine* of an arc or angle is the sine of its complement. In like manner, the *coversed sine*, *cotangent*, and *cosecant* of an arc or angle, are respectively the versed sine, tangent, and secant of its complement.

The sine, versed sine, tangent, and secant may be denoted by the abbreviated expressions, *sin*, *versin* (*or vs.*), *tan*, and *sec*; and the cosine, coversed sine, cotangent, and cosecant, by *cos*, *coversin* (or *covs*), *cotan*, and *cosec*. For the sake of simplicity, the radius of the circle employed for comparing different angles, is generally taken in investigations as unity: when this is not done, it is denoted by its initial letter R.

The sides of a triangle are often conveniently denoted by the small letters corresponding to the capital ones placed at the opposite angles. Thus, *a* denotes the side opposite to the angle A, &c. To prevent ambiguity, we may read A, B, C, *angle* A, *angle* B, *angle* C: while *a*, *b*, *c*, may be called *side a*, *side b*, *side c*.

To illustrate the foregoing definitions, let C be the centre of a circle, and AB, DE two diameters perpendicular to each other. Through any point F in the circumference draw the diameter FL; draw FG perpendicular to AB, and FI to DE; through A draw AH perpendicular to AB, and therefore (III. 16, cor.) touching the circle in A; and let it meet LF produced in H; and, lastly, draw DK perpendicular to DE meeting FL produced in K. Then the arc AF contains the same number of degrees, &c. as the angle ACF; and FG is the sine of this arc and angle; FI, or its equal CG, their cosine; AG their versed sine, and DI their coversed sine; AH their tangent, and CH their secant; DK their cotangent, and CK their cosecant.



From these definitions we derive the following corollaries.

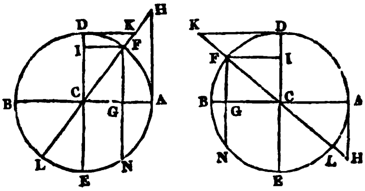
Cor. 1. The sine of an arc AF is half the chord of its double.

For if FG be produced to meet the circumference in N, FN is bisected (III. 3) in G, and (III. 30) the arc FAN in A.

Cor. 2. The sine of the quadrantal arc AD, or of the right angle ACD, is the radius CD.

Cor. 3. If AF be half of AD, and consequently ACF half a right angle, the tangent AH is equal to the radius. For A being a right angle, H must be half a right angle, and (I. 6) AH equal to AC.

Cor. 4. Put the arc AF$=$A, and the radius $=1$. Then (I. 47) $FG^2+CG^2=CF^2$; that is, $\sin^2 A+\cos^2 A=1$. In like manner, we find from the right-angled triangles CAH, CDK, that $CH^2=CA^2+AH^2$, and $CK^2=CD^2+DK^2$; that is, $\sec^2 A=1+\tan^2 A$, and $\operatorname{cosec}^2 A=1+\cot^2 A$.

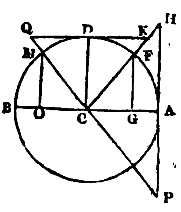


Cor. 5. In the similar triangles CGF, CAH, CG : CF, or CA :: CA : CH; that is, the cosine of an arc is to the radius, as the radius to its secant. Hence also (VI. 17) $CG.CH=CA^2$; that is, $\cos A \sec A=1$. It would be found in like manner from the triangles CIF, CDK, that $\sin A \operatorname{cosec} A=1$, CI being equal to the sine FG.

Cor. 6. In the same triangles CGF, CAH, the cosine CG is to the sine GF, as the radius CA to the tangent AH; whence (VI. 16) $\cos A \tan A=\sin A$. The triangles CIF, CDK give in like manner $\sin A \cot A=\cos A$.

Cor. 7. The radius is a mean proportional between the tangent of an arc and its cotangent. For the triangles CAH, CDK are similar; and therefore HA : AC :: CD, or CA : DK. Hence (VI. 17) $\tan A \cot A=1$.

Cor. 8. The sine of an arc or angle, and the sine of its supplement are equal. So likewise are their cosines, tangents, cotangents, secants, and cosecants. Let AF be an arc, FG, AH its sine and tangent, and CG, DK its cosine and cotangent. Make the angle BCM equal to ACF; draw the perpendicular MO; and produce MC both ways to meet HA, KD produced in P and Q. Then (I. def. 38, and I. 13) the angles BCM, ACM, or ACF, ACM are supplements of each other; as are also the arcs BM, AM, or AF, AM, since (III. 26) BM, AF are equal. Now the triangles CGF, COM are equiangu-



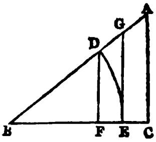
lar, and have the sides CF, CM equal; therefore (I. 26) MO is equal to FG, and CO to CG; and MO, FG are the sines of AM, AF, and CO, CG their cosines. Again, the triangles ACP, ACH are equiangular, and have AC common; therefore (I. 26) AP is equal to AH, and CP to CH; and AP, AH are the tangents, and CP, CH the secants of AM, AF. In like manner it would be proved, by means of the triangles CDQ, CDK, that DQ, the cotangent of AM, is equal to DK, the cotangent of AF, and that their cosecants CQ, CK are equal.

PROPOSITIONS.

Prop. I. Theor.—In a right-angled triangle, the hypotenuse is to either of the legs, as the radius to the sine of the angle opposite to that leg, or to the cosine of the adjacent angle: (2) either of the legs is to the other, as the radius to the tangent of the angle opposite to the latter: and (3) either of the legs is to the hypotenuse, as the radius to the secant of the contained angle.

Let ABC be a triangle, right-angled at C; then (1) $c : b :: \mathrm{R} : \sin \mathrm{B}$, or $\cos \mathrm{A}$; (2) $a : b :: \mathrm{R} : \tan \mathrm{B}$; and (3) $a : c :: \mathrm{R} : \sec \mathrm{B}$.

From B as centre, with any radius, describe an arc cutting AB, BC in D, E; and through D, E draw (I. 12, and 11) DF, EG perpendicular to BC. Then (Trig. def. 5 and 6) FD, EG, and BG are respectively the sine, tangent, and secant of the arc DE, or of the angle B: and, since C is a right angle, A and B (Trig. def. 4) are complements of each other; and therefore (Trig. def. 7) $\sin \mathrm{B} = \cos \mathrm{A}$. Again, since the angle B is common to the triangles ABC, DBF, GBE, and the angles at C, F, E right angles, these triangles (I. 32, cor. 5) are equiangular.



Hence (VI. 4) in the triangles ABC, DBF,

BA : AC :: BD : DF; that is, $c : b :: \mathrm{R} : \sin \mathrm{B}$, or $\cos \mathrm{A}$.

Again, (VI. 4) in the triangles ABC, GBE,

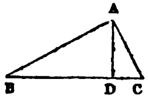
BC : CA :: BE : EG; that is, $a : b :: \mathrm{R} : \tan \mathrm{B}$; and

BC : BA :: BE : BG; that is, $a : c :: \mathrm{R} : \sec \mathrm{B}$.

Cor. Hence (VI. 16) $\mathrm{R}b = c \sin \mathrm{B} = c \cos \mathrm{A}$; that is, *the product of either leg and the radius is equal to the product of the hypotenuse and the sine of the angle opposite to that leg, or of the hypotenuse and the cosine of the adjacent angle.* When $\mathrm{R}=1$, this becomes simply $b = c \sin \mathrm{B} = c \cos \mathrm{A}$.

Prop. II. Theor.—The sides of a plane triangle are proportional to the sines of the opposite angles.

Let ABC be any triangle; then $a : b :: \sin \mathrm{A} : \sin \mathrm{B}$; $a : c :: \sin \mathrm{A} : \sin \mathrm{C}$; and $b : c :: \sin \mathrm{B} : \sin \mathrm{C}$.

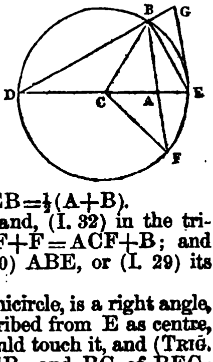


Draw AD perpendicular to BC: then AD is a leg of each of the right-angled triangles ADB, ADC; and therefore (Trig. 1, cor.) R.AD = AB sin B, and R.AD = AC sin C. Hence (I. ax. 1) AB sin B = AC sin C, or c sin B = b sin C: whence (VI. 16) $b : c :: \sin B : \sin C$; and, by drawing perpendiculars from B and C to the opposite sides, it would be proved in a similar manner, that $a : c :: \sin A : \sin C$, and $a : b : \sin A : \sin B$. See the Notes at the end of the work.

Schol. From one of the foregoing analogies, we have, by inversion, $c : b :: \sin C : \sin B$. If C be a right angle, this (Trig. def. cor. 2) becomes $c : b :: R : \sin B$, as in Prop. 1. The first part, therefore, of that proposition is a particular case of this one.

Prop. III. Theor.—The sum of any two sides of a triangle is to their difference, as the tangent of half the sum of the angles opposite to those sides, is to the tangent of half their difference.

Let ABC be a triangle, a, b any two of its sides, of which a is the greater, and A, B the angles opposite to them: then $a+b : a-b :: \tan \frac{1}{2}(A+B)$: $\tan \frac{1}{2}(A-B)$.



From C as centre, with the greater side a as radius, describe the circle DBE, cutting AC produced in D and E, and BA produced in F; join BD, BE, CF; and draw EG parallel to AB, meeting DB produced in G.

Then because DC and CE are each equal to a, DA is equal to $a+b$, and AE to $a-b$.

Also (I. 32) the exterior angle DCB is equal to A+B: and DEB, which is at the circumference, is (III. 20) half of DCA, which is at the centre; therefore DEB $= \frac{1}{2}(A+B)$.

Again, (I. 5) the angle F is equal to B; and, (I. 32) in the triangle ACF, the exterior angle A = ACF + F = ACF + B; and consequently, ACF = A − B; and (III. 20) ABE, or (I. 29) its equal BEG $= \frac{1}{2}(A-B)$.

Now, since (III. 31) EBD, being in a semicircle, is a right angle, as also (I. 13) EBG; if a circle were described from E as centre, with EB as radius, DBG (III. 16, cor.) would touch it, and (Trig. def. 6) DB would be the tangent of DEB, and BG of BEG; and therefore DB, BG will evidently be proportional to the tangents of those angles to any other radius.*

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* Or strictly, EB : BD :: 1 : tan DEB (Trig. 1) and (inver.) BD : EB :: tan DEB : 1. Also (Trig. 1) EB : BG :: 1 : tan BEG. Hence, ex æquo, BD : BG :: tan DEB : tan BEG.

Lastly, since BA (const.) is parallel to GE, we have (VI. 2)

$$DA : AE :: DB : BG; \text{ that is}$$
$$a+b : a-b :: \tan\tfrac{1}{2}(A+B) : \tan\tfrac{1}{2}(A-B).$$

Prop. IV. Theor.—In a plane triangle, the cosine of half the difference of any two angles, is to the cosine of half their sum, as the sum of the opposite sides to the third side: and (2) the sine of half the difference of any two angles is to the sine of half their sum, as the difference of the opposite sides to the third side.

Let ABC (see the last proposition) be any plane triangle:

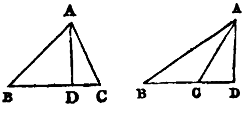
$$\text{then } \cos\tfrac{1}{2}(A-B) : \cos\tfrac{1}{2}(A+B) :: a+b : c;$$
$$\text{and } \sin\tfrac{1}{2}(A-B) : \sin\tfrac{1}{2}(A+B) :: a-b : c.$$

For it was shown in the preceding proposition, that BED $= \frac{1}{2}(A+B)$, and ABE $= \frac{1}{2}(A-B)$; and since DBE is a right angle, DBA is the complement of ABE, and D of BED. But (Trig. 2) in the triangle ABD, sin ABD : sin D :: AD : AB; that is, (Trig. def. 7) $\cos\frac{1}{2}(A-B) : \cos\frac{1}{2}(A+B) :: a+b : c$. Again, (Trig. 2) in the triangle ABE, sin ABE : sin AEB :: AE : AB; that is, $\sin\frac{1}{2}(A-B) : \sin\frac{1}{2}(A+B) :: a-b : c$.

Prop. V. Theor.—In any plane triangle, the sum of the segments of the base made by a perpendicular from the vertex, is to the sum of the other sides, as the difference of those sides to the difference of the segments.

Let ABC be a triangle, and AD a perpendicular from the vertex to the base; the sum of the segments BD, DC is to the sum of the sides AB, AC, as the difference of AB, AC to the difference of BD, DC.

For (II. 5, cor. 4) the rectangle under the sum and difference of AB, AC is equal to the rectangle under the sum and difference of BD, DC; and therefore (VI. 16) the sum of BD, DC is to the sum of AB, AC, as the difference of AB, AC to the difference of BD, DC.



Schol. If the perpendicular fall within the triangle, the segments make up the base, and their difference is less than the base; but if the perpendicular fall without the triangle, as it does (second fig.) when one of the angles at the base is obtuse, the base is the difference of the segments, and their sum is greater than the base.

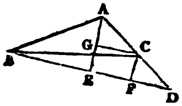
Lemma.—Let $2s$ be put to denote the sum of any three magnitudes a, b, c: then $s-a = \frac{1}{2}(b+c-a)$, $s-b = \frac{1}{2}(a+c-b)$, and $s-c = \frac{1}{2}(a+b-c)$.

For, since $2s = a+b+c$, by subtracting successively $2a$, $2b$, $2c$, and halving the remainders, we obtain those expressions.

PROP. VI. THEOR.*—**The rectangle under two sides of a triangle is to the rectangle under the excesses of half the perimeter above those sides, as the square of the radius to the square of the sine of half the contained angle.**

Let ABC be a triangle, and let $s=\frac{1}{2}(a+b+c)$: then $bc:(s-b)(s-c)::R^2:\sin^2\frac{1}{2}A$.

Produce the less side AC through C, making AD equal to AB; join BD; and draw AE, CF perpendicular, and CG parallel, to BD: then (I. 47, cor. 5) AE bisects BD and the angle A. Now (II. 5, cor. 4) the rectangle under the sum and difference of BC, CD, is equal to the rectangle under the sum and difference of BF, FD, that is, under BD and twice EF: therefore the rectangle under half the sum and half the difference of BC, CD is equal to the rectangle BE.EF. But (TRIG. 1)



$$AB:BE::R:\sin\tfrac{1}{2}A, \text{ and}$$

$$AC:CG \text{ or } EF::R:\sin\tfrac{1}{2}A; \text{ whence (V. Sup. 15)}$$

$$AC.AB:BE.EF, \text{ or } \tfrac{1}{2}(BC+CD).\tfrac{1}{2}(BC-CD)::R^2:\sin^2\tfrac{1}{2}A; \text{ or}$$

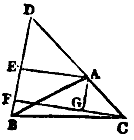
$$bc:\tfrac{1}{2}(a+c-b).\tfrac{1}{2}(a+b-c)::R^2:\sin^2\tfrac{1}{2}A,$$

because $CD=c-b$: and, by the lemma, this becomes $bc:(s-b)(s-c)::R^2:\sin^2\frac{1}{2}A$.

Cor. Hence, taking $R=1$, dividing the product of the means by the first extreme, and extracting the square root, we find $\sin\frac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}$, and it is plain that we should find in a similar manner, $\sin\frac{1}{2}B=\sqrt{\frac{(s-a)(s-c)}{ac}}$, and $\sin\frac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{ab}}$

PROP. VII. THEOR.—**The rectangle under two sides of a triangle is to the rectangle under half the perimeter and its excess above the third side, as the square of the radius, to the square of the cosine of half the angle contained by the two sides.**

Let ABC be a triangle, and let $s=\frac{1}{2}(a+b+c)$: then $bc:s(s-a)::R^2:\cos^2\frac{1}{2}A$.



Produce the less side CA, through A, making AD equal to AB; join BD; draw AE, CF perpendicular, and AG parallel to BD. Then BD (I. 47, cor. 5) is bisected in E; and the angle BAC being (I. 32) equal to the two equal angles D and ABD, each of them is equal to half the angle BAC, that is, half the angle A in the triangle ABC: and (I. 29) GAC is equal to D. Now, it would be shown, as in the preceding proposition, that the rectangle under half the sum and half the difference of DC, CB is equal to the rectangle BE.EF. But (APP. III. 1) $AB:BE::R:\cos ABE$, or $\cos\frac{1}{2}A$; and $AC:AG$, or $EF::R:\cos GAC$, or $\cos\frac{1}{2}A$; whence (V. SUP. 15).

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* For this proposition and the next, much easier proofs than the ones here given will be found in the Author's Elements of Trigonometry, pages 19 and 20. Those proofs are inadmissible here, as they require the use of principles not established by Euclid.

$$AC.AB : BE.EF, \text{ or } \tfrac{1}{2}(DC + CB).\ \tfrac{1}{2}(DC - CB) :: R^2 : \cos^2\tfrac{1}{2}A;$$
$$\text{or } bc : \tfrac{1}{2}(a+b+c).\ \tfrac{1}{2}(b+c-a) :: R^2 : \cos^2\tfrac{1}{2}A,$$

because $DC = b + c$. Hence, by the lemma,

$$bc : s(s-a) :: R^2 : \cos^2\tfrac{1}{2}A.$$

Cor. 1. Hence we find, as in the corollary to the preceding proposition, that $\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$; and it would be proved in a similar manner, that

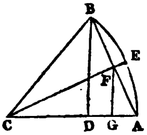
$$\cos\tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}, \text{ and } \cos\tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}.$$

Cor. 2. From the sixth corollary to the definitions of this book, it is plain, that when the radius is unity, if the sine of an arc be divided by its cosine, the quotient is its tangent. Hence, by dividing the expression for the sine of $\frac{1}{2}A$ in the corollary to the preceding proposition, by the value of its cosine in the last corollary, we obtain $\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$; and we should obviously find in a similar manner, that $\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$, and $\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$.

Cor. 3. By dividing the values of $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$, in the preceding corollary, each by that of $\tan\frac{1}{2}A$, we obtain $\frac{\tan\frac{1}{2}B}{\tan\frac{1}{2}A} = \frac{s-a}{s-b}$, and $\frac{\tan\frac{1}{2}C}{\tan\frac{1}{2}A} = \frac{s-a}{s-c}$.

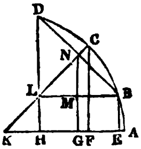
PROP. VIII. PROB.*—Given the radius of a circle, and the cosine of an arc, less than a quadrant; to compute the cosine of half the arc.

Let CD, the cosine of the arc AEB, less than a quadrant, be given; it is required to compute the cosine of its half.



Draw the chord AB, and perpendicular to it draw CFE; draw also FG parallel to BD: then (III. 3, and TRIG. defs. 5 and 7) AF or FB is the sine, and CF the cosine of AE, the half of AEB. Also, (VI. 2) DG is equal to GA, since BF is equal to FA; and $DA = 2DG$; to each of these add 2CD; then $CA + CD = 2CG$, and consequently $CG = \frac{1}{2}(CA+CD)$; or, if the radius be taken as unity, and the arc AEB be denoted by A, $CG = \frac{1}{2}(1 + \cos A)$. Again in the similar triangles ACF, CFG, AC : CF :: CF : CG; whence (VI. 17) $CF^2 = AC.CG$; that is, $\cos^2\frac{1}{2}A = \frac{1}{2}(1 + \cos A)$. Hence, *to compute* $\cos\frac{1}{2}A$, *add* 1 *to* cos A, *take half the sum, and extract the square root.*

PROP. IX. THEOR.—If A and B be any two arcs, R : cos B :: sin A : ½ sin (A − B) + ½ sin (A + B).



Make AC equal to A, and BC, CD each equal to B; draw BE, CF, DH, the sines of AB, AC, AD; join BD, and through the centre K draw KNC: then KN is evidently the cosine of BC. Draw also BML, NMG parallel to AK, DH. Now, in the similar triangles DLB, NMB, since DB is double of NB, DL (VI. 4) is double of NM; to DL add LH, BE, and to 2NM add what is equivalent, 2MG; then $DH + BE = 2NG$; wherefore NG is equal to half the sum of BE and DH, that is, to $\frac{1}{2}\sin(A - B) + \frac{1}{2}\sin(A + B)$. Again, in the similar triangles CFK, NGK, we have (VI. 4, and alternately) CK : NK :: CF : NG; that is, R : cos B :: sin A : $\frac{1}{2}\sin(A - B) + \frac{1}{2}\sin(A + B)$.

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* This proposition and the two following afford means by which trigonometrical tables may be calculated. Much easier methods, however, will be found in the Author's Introduction to the Differential and Integral Calculus, Section V.

Cor. 1. Hence, if $R = 1$, we have, by doubling the second and fourth terms, and by taking the products of the extremes and means, $\sin(A - B) + \sin(A + B) = 2 \cos B \sin A$; whence, $\sin(A + B) = 2 \cos B \sin A - \sin(A - B)$.

Cor. 2. If $B = A$, the last expression becomes simply $\sin 2A = 2 \sin A \cos A$.

PROP. X. PROB.—To investigate a method of computing tables of sines, tangents, and secants.

The chord of 60° (IV. 15, cor.) is equal to the radius 1, and therefore (TRIG. defs. cor. 1) the sine of 30° is 0·5; and (TRIG. defs. cor. 4) $\cos 30° = \sqrt{(1 - \sin^2 30°)}$; whence cos30° may be computed. Then, (TRIG. 8) from the cosine of 30°, we can compute that of 15°; from the cosine of 15°, that of 7° 30′; and thus we may proceed as far as we please, by continual bisections of the arc. By eleven such operations, we should find the cosine of 52″·734375: and its sine would then be computed by means of the fourth corollary to the definitions of this book. Now (as will be proved, APP. I. 40, schol.) a polygon may be inscribed in a circle with so many sides, that its perimeter will differ from the circumference by a quantity as small as we please; and hence it follows, that a very small arc and its sine will have to one another very nearly the ratio of equality, and consequently that the sines of very small arcs are very nearly as the arcs themselves. Hence, therefore, as 52″·734375 to 60″, so is the sine of 52″·734375 to the sine of 60″, or one minute.

The sine of 1′ being thus obtained, take B, in the last formula in the first corollary to the preceding proposition, constantly equal to 1′, and A successively equal to 1′, 2′, 3′, &c. and there will be obtained the following expressions, by means of which the sines of the succeeding minutes may be computed:—

$$\sin 2' = 2 \cos 1' \sin 1',$$
$$\sin 3' = 2 \cos 1' \sin 2' - \sin 1',$$
$$\sin 4' = 2 \cos 1' \sin 3' - \sin 2',$$
$$\sin 5' = 2 \cos 1' \sin 4' - \sin 3',$$

&c. &c. &c.

By this method the sines of all the successive minutes of the quadrant, may be computed: and they comprehend also the cosines; sin 1′, sin 2′, &c. being cos 89° 59′, cos 89° 58′, &c. Then, (TRIG. defs. cor. 5 and 6) the tangents of 1′, 2′, 3′, &c. will be found by dividing the sine of each by its cosine, and their secants by dividing 1 by their cosines.

The numbers which are computed in the manner now explained, are called *natural* sines, tangents, &c. to distinguish them from their logarithms, which are called *logarithmic* sines, tangents, &c.* By the nature of logarithms, the sum of the logarithms of two numbers is equal to the logarithm of their product; and, conversely, if the logarithm of one number be taken from the logarithm of another, the remainder is the logarithm of the quotient obtained by dividing the latter number by the former. Hence, the logarithmic sines, &c. are generally employed in practice; as by their means multiplication and division are exchanged for the easier operations of addition and subtraction. In these tables, the radius is taken for convenience, as 10,000,000,000; and therefore all the sines, tangents, &c. to the radius 1, are multiplied by this number, and the logarithm of the radius is 10.

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* In trigonometrical tables, the sines are generally placed, for convenience, in one column, and the cosines in the adjoining one, or at least in the same page. Now it is plain that the sine is one leg, and the cosine another, of a right-angled triangle, of which the radius is the hypotenuse; and hence the radius, sine, and cosine, may be regarded merely as numbers proportional to the three sides of a right-angled triangle, having one of its angles equal to that to which the sine and cosine belong: and the same is the case respecting the radius, tangent, and secant. Hence trigonometrical tables may be regarded as a register, exhibiting the ratios of the sides of a great number of right-angled triangles, in connexion with the magnitudes of their oblique angles.

Schol. 1. The principles that have been thus established, enable us to solve all the elementary cases of plane trigonometry. Now, of the three sides and three angles of a triangle, some three, and those not the three angles, must be given to determine the triangle;[*] and the resolution of plane triangles may therefore be reduced to the three following cases:—

I. When a side and the opposite angle, and either another side, or another angle, are given:

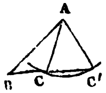
II. When two sides and the contained angle are given:

III. When the three sides are given.

The FIRST CASE is solved on the principle (TRIG. 2) that the sides are proportional to the sines of the opposite angles. Thus, if A, B, a be given, add A and B together, and take the sum from 180°; the remainder (I. 32) is C. Then b and c will be found by the following analogies: $\sin A : \sin B :: a : b$; and $\sin A : \sin C :: a : c$.

If, again, a, b, A be given, we compute B by the analogy, $a : b :: \sin A : \sin B$. Then, C is found by subtracting the sum of A and B from 180°, and c by the analogy, $\sin A : \sin C :: a : c$.

When in this case two unequal sides, and the angle opposite to the less, are given, the angle opposite to the greater (TRIG. defs. cor. 8) may be either that which is found in the table of sines or its supplement; and thus the problem admits of two solutions. To illustrate this, suppose the angle B, the side BA, and the side AC, less than BA, to be given: then, AB and the angle B being made of the given magnitude, if an arc described from A as centre, with the other given side as radius, cut the side opposite to the angle A, in one point C, it will cut it also in another C′; therefore each of the triangles ABC, ABC′ answers equally to the data, and (I. 13) the angle ACB is the supplement of ACC′, or (I. 5) its equal AC′C.



If in this case one of the angles be a right angle, the solution is rather easier; as, by the second corollary to the definitions, the sine of that angle is equal to the radius. The same conclusion may also be obtained by means of the first proposition of this book.[†]

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[*] See the note to the 26th proposition of the first book.

[†] To exemplify the solution of this case, let $a = 13$ yards, $b = 15$ yards, and $A = 53° 8′$,—to resolve the triangle; and the operation by means of logarithms will be as follows:—

As a	13	1·113943	As sin A		9·903108	As sin A		9·903108
: b	15	1·176091	: sin C	59° 29′	9·935246	: sin C	14° 15′	9·391206
: : sin A	53° 8′	9·903108	: : a		1·113943	: : a		1·113943
: sin B	67° 23′, or 112° 37′	9·965256	: c	14	1·146081	: c	4	0·602041

In these operations, to find the fourth term, the second and third terms are added together, and the first is taken from the sum. This may be done very easily, in a single operation, by adding the figures of the second and third terms successively to what remains after taking the right hand figure of the first term from 10, and each of the rest from 9, and rejecting 10 from the final result. Thus, in the first operation, we have 8 and 1 are 9, and 7 are 16; then 1 and 9 are 10 and 5 are 15, &c. It is

The SECOND CASE is solved by means of the third and fourth propositions of this book. Thus, if a, b, C be taken, take C from 180°, and (I. 32) the remainder is A+B. Take the half of this, and then, by the third proposition, as $a+b : a-b :: \tan\frac{1}{2}(A+B) : \tan\frac{1}{2}(A-B)$. This analogy gives half the difference of A and B; and* by adding this and $\frac{1}{2}(A+B)$ together, A, the greater angle, is obtained, while B is found by taking $\frac{1}{2}(A-B)$ from $\frac{1}{2}(A+B)$. The remaining side will be calculated (TRIG. 4) by means of either of the following analogies; and by employing both, an easy verification of the process is obtained:

$$\text{as } \cos\tfrac{1}{2}(A-B) : \cos\tfrac{1}{2}(A+B) :: a+b : c; \text{ and}$$
$$\sin\tfrac{1}{2}(A-B) : \sin\tfrac{1}{2}(A+B) :: a-b : c.$$

When the given angle C is a right angle, the solution is most easily effected by means of the first proposition of this book; the oblique angles being obtained by the analogy, $a : b :: R : \tan B$, or $\cot A$; and the hypotenuse either by the analogy, $R : \sec B :: a : c$, or $\sin A : R :: a : c$.†

The THIRD CASE may be solved by means of the fifth, sixth, or seventh proposition. Thus, assuming a (see the figures for the fifth proposition) as base, we have a to $b+c$ as $b-c$, or $c-b$ to a fourth proportional. If this be less than BC, it is the difference of the segments BD, DC, in the first figure; and if half of it and half of the base be added together, the sum will be the greater segment, while the less will be found by taking half that proportional from half the base. If the fourth proportional be greater than the base, it is the sum of the segments in the second figure, and, as before, the segments are the sum and difference of half the proportional and half the base. Then, by resolving, by the first case, the two right-angled triangles ADB, ADC, in which there are given the

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still easier, however, when the quantity to be subtracted is a sine, to use the cosecant, and when it is a cosine, to use the secant, each diminished by 10, and then to add all the terms together. The reason of this is evident from the nature of logarithms, and from the fifth corollary to the definitions of this book. In like manner, when the number to be subtracted is a tangent, or cotangent, we may use in the former case a cotangent—in the latter a tangent, subtracting in each case 10, either at first or afterwards.

This example evidently belongs to the doubtful case; and hence we have two values for each of the quantities B, C, and c; and therefore two analogies are requisite for finding the values of c.

* See the note to the 13th proposition of the second book.

† As an example, let $a = 57.38$ miles, $b = 42{\cdot}6$ miles, and $C = 56° 45'$; to resolve the triangle.

As $a+b$	99·98	1·999913
: $a-b$	14.78	1·169674
:: $\tan\frac{1}{2}(A+B)$	61° 37′½	10·267498
: $\tan\frac{1}{2}(A-B)$	15° 18′½	9·437259

Hence A = 76° 56′, and B = 46° 13′

As $\cos\frac{1}{2}(A-B)$	15° 18′½	9·984311
: $\cos\frac{1}{2}(A+B)$		9·676913
:: $a+b$		1·999913
: c	49·26	1·692515

As $\sin\frac{1}{2}(A-B)$		9·421626
: $\sin\frac{1}{2}(A+B)$		9·944411
:: $a-b$		1·169674
: c	49·26	1·692459

Half the difference of A and B is here taken as 15° 18′½. When determined accurately, however, it is found to be 15° 18′ 23″. Hence the cause of the slight difference in the logarithm of c, as obtained by the two different analogies. It is plain, that after A and B are computed, c might also be found by means of the first case, by either of the analogies: $\sin A : \sin C :: a : c$, and $\sin B : \sin C :: b : c$. The foregoing method, however, is much preferable.

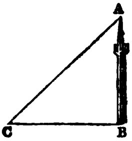
hypotenuses AB, AC, and the legs BD, CD, the angles B, and ACD will be obtained, which, in the first figure are two of the required angles; while in the second, the angle C is the supplement of ACD.

Again, by adding the three sides together, and taking half the sum, the value of *s* is obtained: and, in applying the sixth proposition, the sides containing the required angle are to be separately taken from *s*; but, in applying the seventh, only the side opposite to the required angle is to be subtracted; while if all the three sides be subtracted successively, another mode of solution is furnished by the second and third corollaries to the seventh proposition. This last method is preferable to any of the others, when it is necessary to determine all the angles; and if they be all computed by means of it, the correctness of the operation is ascertained by trying whether their sum is 180°.*

Schol. 2. The following problems exhibit some easy and useful applications of plane trigonometry.

PROB. I.—Let it be required to find the height of an *accessible* object AB, standing on a horizontal plane.

On the horizontal plane take a station C, and measure with a line, a chain, or any such instrument, the distance CB to the base of the object; and with a quadrant, a theodolite, or other angular instrument, measure the angle BCA, called the *angle of elevation*. Then, since B is a right angle, the height AB will be found (TRIG. 1) by the analogy, R : tan C :: CB : BA.



This gives the height of A above CB, the horizontal line passing through the eye of the observer: and therefore to find the entire height, AB must be increased by the height of his eye above the base of the object. The like addition must be made in every problem

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* To exemplify the last of these methods, let $a = 679$, $b = 537$, and $c = 429$; to compute the angles.

Here, by adding the three sides together, we obtain 1645, the half of which, 822·5 is s. Then, by taking from this the three sides successively, we find $s-a = 143·5$, $s-b = 285·5$, and $s-c = 393·5$. The rest of the operation, the subtraction in the first part of which may be performed in the manner pointed out in the example for the first case, is as follows:—

s	822·5	2·915136	subt.
$s-a$	143·5	2·156852	
$s-b$	285·5	2·455606	
$s-c$	393·5	2·594945	
	2)	19·978563	
tan ½A 44° 17′½		9·989281	
A = 88° 35′			

tan ½A	9·989281	add
log $(s-a)$	2·156852	
	12·146133	subt.
log $(s-b)$	2·455606	
tan ½B 26° 7′½	9·690527	
B = 52° 15′		

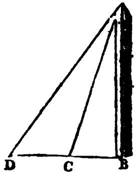
	12·146133	subs.
log $(s-c)$	2·594945	
tan ½C 19° 35′	9·551188	
C = 39° 10′		

In the first part of this operation, the halving of the logarithm serves for the extraction of the square root. The remainder of the work consists in adding together tan½A and log $(s-a)$, and subtracting log $(s-b)$, and log $(s-c)$ successively from of this kind, when the angle of elevation above the horizontal line is given.

PROB. II.—To find the height of an object AB, standing on a horizontal plane, but *inaccessible* on account of the unevenness of the ground near its base, or the intervention of obstacles.

In a straight line passing through the base of the object take two stations C, D; and measure CD, and the two angles of elevation BCA, BDA. Then (I. 32) CAD is the difference of ACB, ADB; and (APP. III. 2) sin CAD : sinD :: DC : CA. Again, (TRIG. 1) R : sinACB :: AC : AB; whence AB will be found.

The computation will be rendered rather more easy by multiplying together (V. SUP. 15) the terms of the two analogies, and dividing the third and fourth terms of the result by AC; as by this means we get the analogy R × sinCAD : sinD× sin BCA :: DC : AB. Hence, to find the logarithm of AB, to the logarithm of DC add the logarithmic sines of D and BCA, and from the sum take the sine of CAD and the radius.

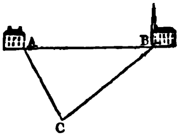


PROB. III.—To find the distance of two objects A and B, on a horizontal plane.

This may be effected in different ways according to circumstances.

1. A base AC may be taken, terminated at one of the objects. The angles A and C, and the side AC are then measured; and the required distance AB is found by the analogy, sinB : sinC :: AC : AB.

2. This method fails, if the objects A and B be not visible from one another, as then the angle A cannot be measured. In this case, a station C may be taken as before, from which both A and B are visible. Then, having measured the angle C, and the sides AC, BC, we compute the distance AB by means of the second case of trigonometry



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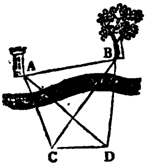
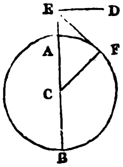
the sum. This method of solution is remarkably easy, requiring for the entire operation only four logarithms to be taken from the tables; and affording at the same time a most satisfactory verification by the addition of the three angles, when found. The preparatory part of the process also admits of an easy verification, as the sum of the three remainders $s-a$, $s-b$, $s-c$ is equal to the half sum.

3. When, from inequalities in the ground, or other causes, the preceding methods are inapplicable, the solution may be effected in the following manner. Measure a base CD, such that A and B are both visible from each of its extremities: measure also the two angles at C, and the two at D. Then, by the first case in trigonometry, we compute AC in the triangle ACD, and BC in the triangle BCD: from which, and from the contained angle ACB, AB is computed by means of the second case. The operation may be verified by computing AD, BD, by the first case, and thence AB by the second case.



PROB. IV.—Let AFB be a great circle of the earth, supposed to be a sphere; E a point in the diameter BA produced, EF a straight line touching the circle in F, and ED a straight line in its plane, perpendicular to AB: it is required to compute the angle DEF, and the straight line EF.

Draw the radius CF. Then, since (III. 18) CFE is a right angle, we have (hyp. and I. 32, cor. 3) DEC = CEF + ECF. Take away CEF, and there remains DEF = ECF. Now (III. 36) $EF^2 =$ BE.EA. Hence EF will be found by adding AE to AB, multiplying the sum by EA, and extracting the square root. To find CE, add AE to the radius AC. Then (TRIG. 1) CE : EF : : R : sinECF, or sinDEF; or CE : CF : : R : cosECF, or cosDEF.



Schol. The foregoing solution gives correct results in every case. When AE, however, is inconsiderable in comparison of AB, a very near approximation to the true results is obtained, in an easy manner, by using BA and CA instead of BE and CE, and the arc AF instead of EF. In this way, putting n equal to the feet in AE, since a mile is equivalent to 5280 feet, we have EF = $\sqrt{(BA.AE)} = \sqrt{(7912 \times 5280 \times n)}$; or, in miles, EF =

$$\frac{\sqrt{(7912 \times 5280 \times n)}}{5280} = \sqrt{\frac{7912 \times n}{5280}} = \sqrt{\frac{3n}{2}},$$

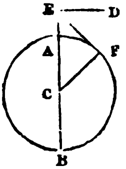
very nearly, since the sum of 5280 and its half is 7920, which differs very little from 7912. Hence, therefore, *to find* EF *in miles, to* AE *in feet, add its half, and extract the square root.*

Again, (APP. I. 40, schol.) the circumference of AFB $= 7912 \times \pi$; and as $7912 \times \pi$: AF, or EF : : 360° : ACF, or DEF; that is, as $7912 \times \pi : \sqrt{\frac{7912 \times n}{5280}} :: 360^\circ$: DEF. Hence, by dividing the first and second terms by 7912, reducing the third to seconds, and dividing the product of the means by the first term, we get, in seconds,

$$\text{DEF} = \frac{360 \times 3600}{\pi} \times \sqrt{\frac{n}{7912 \times 5280}}$$

$$= \frac{360 \times 3600}{\pi\sqrt{(2 \times 7912 \times 5280)}} \times \sqrt{2n} = 90\tfrac{1}{4} \times \tfrac{1}{2}\sqrt{2n},$$

very nearly; as is easily shown by substituting (App. I. 4°, schol.) for π its value 3·141593, and computing the multiplier of $\sqrt{2n}$, which is readily found to be $45\frac{1}{8}$, or $90\frac{1}{4} \times \frac{1}{2}$, nearly. To find DEF, therefore, in seconds, *double* AE in feet, *extract the square root, and multiply half the result by* $90\frac{1}{4}$.



If E be the position of the eye, the angle DEF is called the *dip of the horizon*, and F is a point in the line in which the sky and sea appear to meet, and which is called the *offing*.*

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* Hence, if AE, the height of the eye, be six feet, the distance of the offing will be three miles; if 24 feet, 6 miles; if 32 feet, 7 miles, nearly; if 600 feet, 30 miles; if 3000 feet, 67 miles, nearly, &c. If, again, the height of the eye be eight feet, the dip will be 180″, or 3′, nearly; if 18 feet, 4′ 31″, nearly; if 1000 feet, 33′ 38″, nearly, &c.

BOOK XI.

DEFINITIONS.

1. A STRAIGHT line is said to be *perpendicular* to a plane when it makes right angles with all straight lines meeting it in that plane.

2. The *inclination* of two planes which meet one another is the angle contained by two straight lines drawn from any point of their common section at right angles to it, one upon each plane.*

3. If that angle be a right angle, the planes are *perpendicular* to one another.

4. *Parallel* planes are such as do not meet one another, though produced ever so far in every direction.

5. A *solid* angle is that which is made by more than two plane angles meeting in one point, and not lying in the same plane.

If the number of plane angles be three, the solid angle is *trihedral;* if four, *tetrahedral;* if more than four, *polyhedral.*

6. A *polyhedron* is a solid figure contained by plane figures.

If it be contained by four plane figures, it is called a *tetrahedron;* if by six, a *hexahedron;* if by eight, an *octahedron;* if by twelve, a *dodecahedron;* if by twenty, an *icosahedron,* &c.

7. A *regular body,* or *regular polyhedron* is a solid contained by plane figures, which are all equal and similar.

8. Of *solid figures contained by planes,* those are *similar,* which have all their solid angles equal, each to each, and which are contained by the same number of similar plane figures, similarly situated.

9. A *pyramid* is a solid figure contained by one plane figure, called its *base,* and by three or more triangles meeting in a point without the plane, called the *vertex* of the pyramid.

10. A *prism* is a solid figure, the ends or *bases* of which are parallel, and are equal and similar plane figures, and its other boundaries are parallelograms. One of these parallelograms also is sometimes regarded as the *base* of the prism.

11. Pyramids and prisms are said to be *triangular,* when their bases are triangles; *quadrangular,* when their bases are quadrilaterals; *pentagonal,* when they are pentagons, &c.

12. The *altitude of a pyramid* is the perpendicular drawn from its vertex to its base: and the *altitude of a prism* is either a perpendicular drawn from any point in one of its ends or bases to the

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* The angle which one plane makes with another, is sometimes called a *dihedral* angle.

other; or a perpendicular to one of its bounding parallelograms from a point in the line opposite. The first of these altitudes is sometimes called the *length* of the prism.

13. A prism, of which the ends or bases are perpendicular to the other sides, is called a *right prism:* any other is an *oblique prism.*

14. A *parallelepiped* is a prism of which the bases are parallelograms.

15. A parallelepiped of which the bases and the other sides are rectangles, is said to be *rectangular.*

16. A *cube* is a rectangular parallelepiped, which has all its six sides squares.

17. A *sphere* is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

18. The *axis* of a sphere is the fixed straight line about which the semicircle revolves.

19. The *centre* of a sphere is the same as that of the generating semicircle.

20. A *diameter* of a sphere is any straight line which passes through the centre, and is terminated both ways by its surface.

21. A *cone* is a solid figure described by the revolution of a right-angled triangle about one of the legs, which remains fixed.

If the fixed leg be equal to the other leg, the cone is called a *right-angled* cone: if it be less than the other leg, an *obtuse-angled,* and if greater, an *acute-angled* cone.

22. The *axis* of a cone is the fixed straight line about which the triangle revolves.

23. The *base* of a cone is the circle described by the leg which revolves.

24. A *cylinder* is a solid figure described by the revolution of a rectangle about one of its sides, which remains fixed.

25. The *axis* of a cylinder is the fixed straight line about which the rectangle revolves.

26. The *bases,* or *ends,* of a cylinder are the circles described by the two revolving opposite sides of the rectangle.

27. *Similar* cones and cylinders are those which have their axes and the diameters of their bases proportionals.

PROPOSITIONS.

PROP. I. THEOR.—One part of a straight line cannot be in a plane, and another part above it.

If it be possible, let AB, a part of the straight line ABC, be in the plane, and the part BC above it: and (I. post. 2) since AB is in the plane, it can be produced in that plane;* let it be pro-

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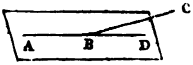
* This depends also on the fifth definition of the first book; which for the sake of this proposition ought to be as follows:—A plane surface is that in which any two points being taken, the straight line passing through them lies wholly in that surface.

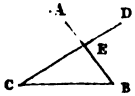
duced to D; and let any plane pass through AD, and be turned about on AD, till it pass through C; and (I. def. 5) because the points B, C are in this plane, BC is in it: therefore, there are two straight lines, ABC, ABD, in the same plane, that have a common segment AB, which (I. def. 3, cor.) is impossible. Therefore, &c.



Prop. II. Theor.—Two straight lines which cut one another are in one plane: so also are three straight lines which meet one another, not in the same point.

Let two straight lines AB, CD, cut one another in E; AB, CD are in one plane: so also are three straight lines EC, CB, BE, which meet one another, not in the same point.

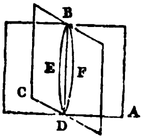
Let any plane pass through EB, produced, if necessary, and let it be turned about EB, till it pass through the point C. Then (I. def. 5) because the points E, C are in this plane, the straight line EC is in it. For the same reason BC is in the same; and (hyp.) EB is in it: therefore EC, CB, BE are in one plane. But in the plane in which EC, EB are, in the same (XI. 1) are CD, AB. Therefore AB, CD are in one plane. Wherefore, &c.



Prop. III Theor.—If two planes cut one another, their common section is a straight line.

Let two planes AB, BC cut one another, and let the line DB be their common section; DB is a straight line.

If it be not, from D to B, draw, in the plane AB, the straight line DEB, and in the plane BC the straight line DFB. Then two straight lines DEB, DFB enclose a space; which (I. def. 3, cor.) is impossible: wherefore BD, the common section of the planes AB, BC, cannot but be a straight line.* Therefore, &c.



Prop. IV. Theor.—If a straight line be perpendicular to each of two straight lines at their point of intersection, it is also perpendicular to the plane in which they are.

Let the straight line EF be perpendicular to each of the straight lines AB, CD at their intersection E: EF is also perpendicular to the plane passing through AB, CD.

Take the straight lines EB, EC equal to one another, and join BC: in BC and EF take any points G and F, and join EG, FB, FG, FC. Then, in the triangles BEF, CEF, BE is equal to

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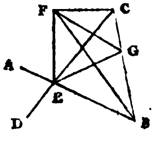
* Thus, when the walls, floor, and ceiling of an apartment are planes, the lines in which they cut each other are straight lines. So likewise are the ridge and the eaves of a house, when the walls and the sloping surfaces of the roof are planes.

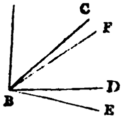
CE; EF common; and the angles BEF, CEF are equal, being (hyp.) right angles: therefore (I. 4) BF is equal to CF. The triangle BFC is therefore isosceles; and (II. 5, cor. 5) the square of BF is equal to the square of FG and the rectangle BG.GC. For the same reason, because (const.) the triangle BEC is isosceles, the square of BE is equal to the square GE and the rectangle BG.GC. To each of these add the square of EF; then the squares of BE, EF are equal to the squares of GE, EF, and the rectangle BG.GC. But (I. 47) the squares of BE, EF are equal to the square of BF, because BEF is a right angle; and it has been shown, that the square of BF is equal to the square of FG and the rectangle BG.GC: therefore the square of FG and the rectangle BG.GC are equal to the squares of GE, EF, and the rectangle BG.GC. Take the rectangle BG.GC from each, and there remains the square of FG, equal to the squares of GE, EF: wherefore (I. 48) FEG is a right angle. In the same manner it would be proved, that EF is perpendicular to any other straight line drawn through E in the plane passing through AB, CD. But (XI. def. 1) a straight line is perpendicular to a plane, when it makes right angles with all straight lines meeting it in that plane: therefore EF is perpendicular to the plane of AB, CD. Wherefore, if a straight line, &c.



PROP. V. THEOR.—If three straight lines meet all in one point, and a straight line be perpendicular to each of them at that point; the three straight lines are in the same plane.

Let the straight line AB be perpendicular to each of the straight lines BC, BD, BE, at B, the point in which they meet; BC, BD, BE are in one and the same plane.

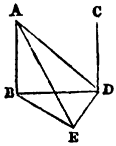
If not, let, if possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the common section of which with the plane in which BD and BE are, is (XI. 3) a straight line; let this be BF. Therefore the three straight lines BA, BC, BF are all in one plane, viz. that which passes through AB, BC; and (XI. 4) because AB is at right angles to each of the straight lines BD, BE, it is also at right angles to every straight line meeting it in that plane; therefore ABF is a right angle: but (hyp.) ABC is also a right angle: therefore the angles ABF, ABC, in the same plane, are equal, which (I. ax. 9) is impossible. Therefore BC is not above the plane of BD and BE: wherefore BC, BD, BE are all in the same plane. Therefore, if three straight lines, &c.



PROP. VI. THEOR.—If two straight lines be at right angles to the same plane, they are parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane BDE; AB is parallel to CD.

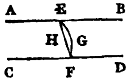
Let them meet the plane in the points B, D; join BD, and draw DE perpendicular to BD, in the plane BDE; make DE equal to AB, and join BE, AE, AD. Then, because AB is perpendicular to the plane, each of the angles ABD, ABE is (XI. def. 1) a right angle. For the same reason, CDB, CDE are right angles. And because AB is equal to DE, BD common, and the angle ABD equal to BDE, AD is equal (I. 4) to BE. Again, in the triangles ABE, ADE, AB is equal to DE, BE to AD, and AE common; therefore (I. 8) the angle ABE is equal to EDA: but ABE is a right angle; therefore EDA is also a right angle, and ED perpendicular to DA: it is also perpendicular to each of the two BD, DC; therefore (XI. 5) these three straight lines DA, DB, DC, are all in the same plane. But (XI. 2) AB is in the plane in which are BD, DA: therefore AB, BD, DC are in one plane. Now (hyp.) each of the angles ABD, BDC is a right angle; therefore (I. 28) AB is parallel to CD. Wherefore, &c.



PROP. VII. THEOR.—If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Let AB, CD be parallel straight lines: the straight line which joins any points E and F in those lines, is in the same plane with the parallels.

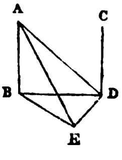
If not, let it be, if possible, above the plane, as EGF; and in the plane ABDC, in which the parallels are, draw (I. post. 2) the straight line EHF from E to F; and since EGF also is a straight line, the two straight lines EHF, EGF include a space between them, which (I. def. 3, cor.) is impossible. Therefore, the straight line joining E, F, is not without the plane in which the parallels AB, CD are: it is therefore in that plane. Wherefore, if two straight lines, &c.



PROP. VIII. THEOR. — If one of two parallel straight lines be perpendicular to a plane, the other is also perpendicular to it.

Let AB, CD be parallel, and let one of them AB be perpendicular to the plane BDE; the other is also perpendicular to BDE.

Let AB, CD meet the plane in the points B, D, and join BD; then (XI. 7) AB, CD, BD are in one plane. In the plane BDE draw DE perpendicular to BD, and equal to AB, and join BE, AE, AD. Then, because AB is perpendicular to the plane, the angles ABD, ABE are (XI. def. 1) right angles. Again, the angles ABD,

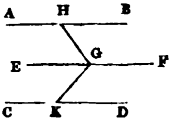


CDB are together equal (I. 29) to two right angles: but ABD is a right angle; therefore also CDB is a right angle, and CD perpendicular to BD. Then it would be proved, as in the sixth proposition of this book, that ED is perpendicular to DA: and being also (const.) perpendicular to BD, it is perpendicular (XI. 4) to the plane passing through DA, DB. But (XI. 2) AB is likewise in this plane; and so also is CD : DE, therefore (XI. def. 1) is perpendicular to CD. Therefore, since CD is perpendicular to each of the two straight lines DB, DE, it is perpendicular (XI. 4) to the plane BDE in which those lines are. Wherefore, if one, &c.

PROP. IX. THEOR.—Two straight lines which are each of them parallel to the same straight line, and are not both in the same plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not both in the same plane with it; AB is parallel to CD.

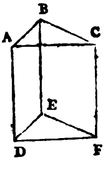
From any point G in EF, draw GH, in the plane passing through EF, AB, at right angles to EF; and in the plane passing through EF, CD, draw GK also at right angles to EF. Then, because EF is perpendicular both to GH and GK, it is perpendicular (XI. 4) to the plane HGK passing through them. But EF is parallel to AB; therefore AB is at right angles (XI. 8) to HGK. For the same reason, CD is likewise at right angles to HGK. Therefore, since AB, CD are each of them at right angles to HGK, AB is parallel (XI. 6) to CD. Wherefore, two straight lines, &c.



Schol. The same has been proved (I. 30) respecting straight lines in the same plane: therefore, universally, straight lines which are parallel to the same straight line, are parallel to one another.

PROP. X. THEOR.—If two straight lines meeting one another, be parallel to two others that meet one another, and are not in the same plane with the first two; the first two and the other two contain equal angles.*

Let the straight lines AB, BC, which meet one another, be parallel to DE, EF, which also meet one another, but are not in the same plane with AB, BC: the angle ABC is equal to DEF.



Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF. Because BA is equal and parallel to ED, therefore AD is (I. 33) both equal and parallel to BE. For the same reason, CF is equal and parallel to BE. Therefore AD and CF being each of them parallel to BE, are (XI. 9) parallel to one another. They are also (I. ax. 1) equal; and AC, DF join them towards the same parts; and therefore (I. 33) AC is equal and parallel to DF. And because AB, BC are equal to DE, EF, and AC to DF; the angle ABC is equal (I. 8) to DEF. Therefore, if two straight lines, &c.

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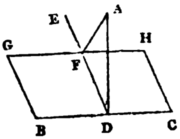
* Or supplemental ones; as will be plain after the demonstration here given, if AB be produced through B. This generalizes the 4th corollary to the 34th proposition of the first book.

PROP. XI. PROB.— To draw a straight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from A a perpendicular to BH.

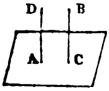
In the plane draw any straight line BC, and (I. 12) from A draw AD perpendicular to BC. Then, if AD be also perpendicular to the plane BH, the thing required is done. But if it be not, from D (I. 11) draw DE, in the plane BH, at right angles to BC; from A draw AF perpendicular to DE; and through F draw (I. 31) GH parallel to BC. Then, because BC is at right angles to ED and DA, BC is at right angles (XI. 4) to the plane passing through ED, DA: and GH being parallel to BC, is also (XI. 8) at right angles to the plane through ED, DA; and it is therefore perpendicular (XI. def. 1) to every straight line meeting it in that plane: GH is consequently perpendicular to AF. Therefore AF is perpendicular to each of the straight lines GH, DE; and consequently (XI. 4) to the plane BH: wherefore AF is the perpendicular required.



PROP. XII. PROB.—To draw a straight line perpendicular to a given plane, from a point given in the plane.

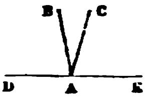
Let A be the point given in the plane; it is required to draw a perpendicular from A to the plane.

From any point B, above the plane, draw (XI. 11) BC perpendicular to it: if this pass through A, it is the perpendicular required. If not, from A draw (I. 31) AD parallel to BC. Then, because AD, CB are parallel, and one of them BC is at right angles to the given plane, the other AD is also (XI. 8) at right angles to it.



PROP. XIII. THEOR.—From the same point in a given plane, there cannot be two straight lines drawn perpendicular to the plane, upon the same side of it: and there can be but one perpendicular to a plane from a point above it.

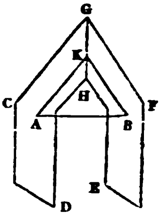
For, if it be possible, let AB, AC, be at right angles to a given plane from the same point A in the plane, and upon the same side of it; and let a plane pass through BA, AC; DAE, the common section of this with the given plane is (XI. 3) a straight line passing through A. Then (XI. def. 1) because BA is at right angles to the given plane, BAE is a right angle. For the same reason CAE is a right angle. Therefore BAE is equal to CAE; which (I. ax. 9) is impossible. Also, from a point above a plane, there can be but one perpendicular to that plane; for, if there could be two, they would be parallel (XI. 6) to one another, which (I. def. 11) is absurd. Therefore, from the same point, &c.



Prop. XIV. Theor.—Planes to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD, EF; these planes are parallel to one another.

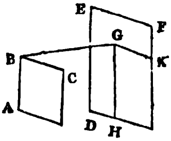
If not, they will meet when produced; let them meet; their common section will be a straight line GH, in which take any point K, and join AK, BK. Then because AB is perpendicular to the plane EF, it is perpendicular (XI. def. 1) to the straight line BK, which is in that plane. Therefore ABK is a right angle. For the same reason, BAK is a right angle; wherefore, in the triangle ABK, the two angles ABK, BAK are equal to two right angles, which (I. 17) is impossible. Therefore the planes CD, EF, though produced, do not meet one another; that is, (XI. def. 4) they are parallel. Therefore planes, &c.



Prop. XV. Theor.—Two planes are parallel, if two straight lines which meet one another on one of them, be parallel to two which meet on the other.

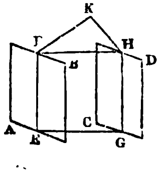
Let the straight lines AB, BC meet on the plane AC, and DE, EF, on the plane DF: if AB, BC be parallel to DE, EF, the plane AC is parallel to DF.

From B draw (XI. 11) BG perpendicular to the plane DF, and let it meet that plane in G; and through G draw (I. 31) GH parallel to ED, and GK to EF. Then, because BG is perpendicular to the plane DF, each of the angles BGH, BGK is (XI. def. 1) a right angle: and because (XI. 9) BA is parallel to GH, each of them being parallel to DE, the angles GBA, BGH are together equal (I. 29) to two right angles. But BGH is a right angle; therefore also GBA is a right angle, and GB perpendicular to BA. For the same reason, GB is perpendicular to BC. Since therefore GB is perpendicular to BA, BC, it is perpendicular (XI. 4) to the plane AC: and (const.) it is perpendicular to the plane DF. But (XI. 14) planes to which the same straight line is perpendicular are parallel to one another: therefore the planes AC, DF are parallel. Wherefore, two planes, &c.



PROP. XVI. THEOR.—If two parallel planes be cut by another plane, their common sections with it are parallels.

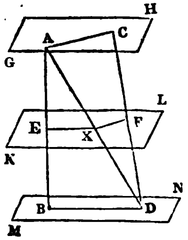
Let the parallel planes, AB, CD, be cut by the plane EFHG, and let their common sections with it be EF, GH; EF is parallel to GH.



For, if it be not, EF, GH will meet if produced, either towards FH or EG. First, let them be produced towards FH, and meet in the point K. Then, since EFK is in the plane AB, K is in AB. For the same reason, K is also in CD: wherefore the planes AB, CD produced meet one another; but they do not meet, since (hyp.) they are parallel. Therefore EF, GH do not meet when produced through F, H: and it may be proved in the same manner, that they do not meet when produced through E, G; they are therefore (I. def. 11) parallel. Wherefore, if two parallel planes, &c.

PROP. XVII. THEOR.—If two straight lines be cut by parallel planes, they are cut in the same ratio.

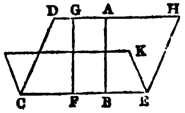
Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D; as AE : EB :: CF : FD.



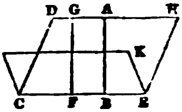
Join AC, BD, AD, and let AD meet KL in X; join also EX, XF. Because the two parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are (XI. 16) parallel. For the same reason, because GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel. Then, (VI. 2) because EX is parallel to BD, a side of the triangle ABD, AE : EB :: AX : XD; and because XF is parallel to AC, a side of the triangle ADC, AX : XD :: CF : FD; and it was proved that AX : XD :: AE : EB; therefore (V. 11) AE : EB :: CF : FD. Wherefore, if two straight lines, &c.

PROP. XVIII. THEOR.—If a straight line be perpendicular to a plane, every plane which passes through it is perpendicular to that plane.

Let AB be perpendicular to the plane CK; every plane which passes through AB is perpendicular to CK.



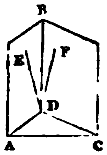
Let any plane DE pass through AB, and let CE be its common section with CK; take any point F in CE, from which draw FG in the plane DE perpendicular to CE. Then, because AB is perpendicular to CK, it is perpendicular (XI. def. 1) to CE, which meets it in that plane. Therefore ABF is a right angle; but (const.) GFB is likewise a right angle: therefore (I. 28) AB is parallel to FG. Now AB is at right angles to the plane CK; therefore (XI. 8) FG is also at right angles to the same plane. But (XI. def. 3) one plane is at right angles to another, when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles to the other plane; and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane CK; therefore DE is at right angles to CK. In like manner, it may be proved, that any other plane passing through AB is perpendicular to CK. Therefore, if a straight line, &c.



Prop. XIX. Theor.—If two planes cutting one another, be each perpendicular to a third plane, their common section is perpendicular to the same plane.

Let the two planes AB, BC be each perpendicular to a third plane ADC, and let BD be the common section of the first two; BD is perpendicular to ADC.

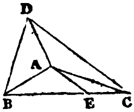
If it be not, from D draw, in AB, the straight line DE perpendicular to AD, the common section of AB with ADC; and in BC draw DF perpendicular to CD, the common section of BC with ADC. Then, because AB is perpendicular to ADC, and DE is drawn in AB at right angles to AD their common section, DE (XI. def. 3) is perpendicular to ADC. In the same manner it may be proved that DF is perpendicular to ADC. Therefore, from the point D two straight lines are drawn at right angles to the third plane, upon the same side of it, which (XI. 13) is impossible: from D, therefore, there cannot be any perpendicular to the third plane, except BD the common section of the planes AB, BC; BD, therefore, is perpendicular to the third plane. Wherefore, if two planes, &c.



Prop. XX. Theor.—If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB: any two of these are greater than the third.

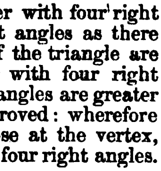
If the angles be all equal, it is evident that any two of them are greater than the third. But if they be not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; and make in the plane of BA, AC, the angle BAE equal (I. 23) to DAB; make AE equal to AD; through E draw BEC cutting AB, AC, in the points B, C, and join DB, DC. Then, in the triangles BAD, BAE, because DA is equal to AE, AB common, and the angle DAB is equal to EAB; DB is equal (I. 4) to BE. Again, because (I. 20) BD, DC are greater than CB, and one of them BD has been proved equal to BE a part of CB, therefore the other DC is greater (I. ax. 5) than the remaining part EC. Then because DA is equal (const.) to AE, and AC common, but the base DC greater than the base EC; therefore (I. 25) the angle DAC is greater than EAC; and (const.) the angles DAB, BAE are equal; wherefore (I. ax. 4) the angles DAB, DAC are together greater than BAE, EAC, that is, than BAC. But BAC is not less than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other. Wherefore, if a solid angle, &c.



Prop. XXI. Theor.—Every solid angle is contained by plane angles, which are together less than four right angles.

Let the solid angle at A be contained by any number of plane angles, BAC, CAD, DAE, EAF, FAB; these together are less than four right angles.

Let the planes in which the angles are, be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB. Then, because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which (XI. 20) any two are greater than the third, CBA, ABF are greater than FBC. For the same reason, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the figure BCDEF. Therefore all the angles at the bases of the triangles are together greater than all the angles of that figure: and because (I. 32) all the angles of the triangles are together equal to twice as many right angles as there are triangles; that is, as there are sides in the figure BCDEF; and that (I. 32, cor. 1) all the angles of the figure, together with four right angles, are likewise equal to twice as many right angles as there are sides in the figure; therefore all the angles of the triangle are equal to all the angles of the figure, together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the figure, as has been proved: wherefore the remaining angles of the triangles, viz., those at the vertex, which contain the solid angle at A, are less than four right angles. Therefore every solid angle, &c.



Schol. This proposition does not necessarily hold, if any of the angles of the rectilineal figure BCDEF be re-entrant; or, which is the same, if any of the planes forming the solid angle at A, being produced, pass through that angle.*

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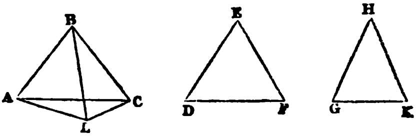
* The reason of this is plain from the note to the second corollary to the 32d proposition of the first book.

Prop. XXII. Theor.—If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal; a triangle may be made, having its sides equal, each to each, to the straight lines that join the extremities of those equal straight lines.

Let ABC, DEF, GHK be three plane angles, every two of which are greater than the third: then, if AB, BC, DE, EF, GH, HK be all equal, and if AC, DF, GK be joined; a triangle may be made of AC, DF, GK; that is, every two of these lines are together greater than the third.

If the angles at B, E, H be equal, AC, DF, GK are also (I. 4) equal; and any two of them are greater than the third.

But if these angles be not all equal, let ABC be not less than either of the two at E, H; therefore (I. 4 and 24) the straight line AC is not less than either of the other two DF, GK; and it is plain that AC, together with either of the other two, must be greater than the third. Also DF and GK are greater than AC. For, make the angle ABL equal to the angle H, and make BL equal to AB, or GH; and join AL, LC. Then, because AB, BL are equal to GH, HK, and the angle ABL to H, the base AL is equal to GK. And, because (hyp.) the angles E, H are greater than the angle ABC, of which H is equal to ABL, therefore the remaining angle E is greater than LBC. Then, because the two sides LB, BC are equal to the two DE, EF, but the angle DEF greater than LBC, DF is greater (I. 24) than LC: and it has been proved that GK is equal to AL: therefore DF and GK are greater than AL and LC. But AL and LC are greater (I. 20) than AC: much more then are DF and GK greater than AC. Wherefore any two of these straight lines AC, DF, GK are greater than the third; and, therefore, a triangle may be made (I. 22) the sides of which will be equal to AC, DF, GK. If every two, therefore, &c.

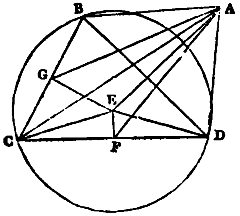


Lemma.—If a solid angle be contained by three planes, and if on the intersections of those planes, three equal lines be taken, terminated in the vertex of the solid angle; each of these lines is greater than the radius of the circle described about the triangle formed by joining their other extremities.

Let the solid angle at A be contained by the three plane angles, BAC, CAD DAB, and let AB, AC, AD be taken all equal; join

BC, CD, DB; about the triangle BCD describe (IV. 5) a circle, and draw the radius EC: EC is less than AC.

Let BC, CD, two sides of the triangle BCD, which do not pass through the centre E, be bisected in G and F, and join EA, ED, EF, EG, AF, AG. Then, since the triangles CAD, CED (hyp. and I. def. 30) are isosceles, and since AF, EF are drawn from their vertices bisecting their common base CD, AFC, EFC are (III. 3, cor. 1) right angles; and therefore (XI. 4) CF is perpendicular to the plane AFE. But the plane BCD passes through CF, therefore (XI. 18) the plane BCD is perpendicular to AEF; and consequently the plane AEF is perpendicular to BCD. It would be proved in the same manner, that the plane AEG is also perpendicular to the plane BCD. Now AE is the common section of the planes AEF, AEG; and since each of these planes is perpendicular to the plane BCD, AE is also (XI. 19) perpendicular to BCD; and therefore (XI. def. 1) AEC is a right angle, and (I. 19) AC is greater than CE. Therefore, if a solid angle, &c.



Cor. Hence, if there be three plane angles B, E, H (see the figures for the next proposition), which are such that a solid angle may be contained by them, and if BA, BC, ED, EF, HG, HK be taken all equal, and AC, DF, GK be drawn: AB is greater than the radius of the circle described about a triangle having its three sides equal to AC, DF, GK, each to each.

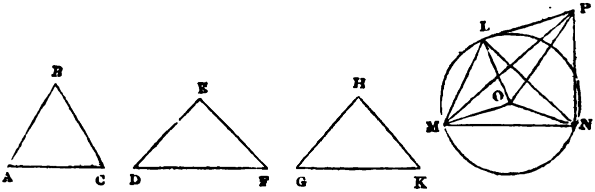
PROP. XXIII. PROB.—To make a solid angle having the angles containing it, equal to three given plane angles, any two of which are greater than the third, and all three together less than four right angles.

Let B, E, H be given plane angles, any two of which are greater than the third, and all of them together less than four right angles: it is required to make a solid angle contained by plane angles equal to B, E, H, each to each.

From the lines containing the angles, cut off BA, BC, ED, EF, HG, HK, all equal to one another; and join AC, DF, GK: then (XI. 22) a triangle may be made of three straight lines equal to AC, DF, GK. Let this (I. 22) be the triangle LMN, AC being equal to LM, DF to MN, and GK to LN.

About LMN describe (IV. 5) a circle, and draw the radii, LO, MO, NO; draw also OP (XI. 12) perpendicular to the plane LMN. Then, by the corollary to the lemma, any of the radii LO, MO, NO is less than AB. Find (I. 47, cor. 3) the side of a square equal to the difference of the squares of AB and LO; make OP equal to that side; and join PL, PM, PN: the plane angles LPM, MPN, and NPL form the solid angle required.

For, since OP is (const.) perpendicular to the plane LMN, the angles, LOP, MOP, NOP, are (XI. def. 1) right angles: and therefore, since, in the triangles OLP, OMP, ONP, the sides OL, OM, ON are equal, OP common, and the contained angles

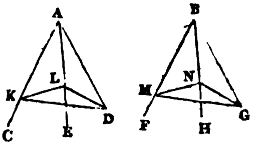


equal, the bases LP, MP, NP are (I. 4) all equal. Also (const.) the square of AB is equal to the squares of LO, OP; and (I. 47) the square of LP is also equal to the squares of LO, OP, because LOP is a right angle. Therefore (I. ax. 1) the square of AB is equal to the square of PL, and (I. 46, cor. 3) AB to PL: and hence all the straight lines LP, MP, NP, BA, BC, ED, &c. are equal. Then, in the triangles LPM, ABC, the sides AB, BC are equal to LP, PM, each to each; and (const.) AC is equal to LM: therefore (I. 8) the angles ABC, LPM are equal; and it would be shown in a similar manner, that the angle E is equal to MPN, and H to NPL. The solid angle at P, therefore, being contained by three plane angles, which are equal to the three given angles, B, E, H, each to each, is such as was required.

Prop. A. Theor.—If two solid angles be each contained by three plane angles, equal to one another, each to each: the planes in which the equal angles are, have the same inclination.

Let there be two solid angles at A and B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three FBG, FBH, HBG, which are equal to the other three, each to each: the planes in which the equal angles are, have the same inclination to one another.

In AC take any point K, and in the plane CAD, from K draw KD at right angles to AC; and in the plane CAE draw KL at right angles to the same AC; therefore (XI. def. 2) the angle DKL is the inclination of the planes CAD, CAE. In BF take BM equal to AK, and from M draw in the planes FBG, FBH, MG, MN, perpendicular to BF; therefore the angle GMN is the inclination of the planes FBG, FBH. Join LD, NG; and because (hyp.) in the triangles KAD, MBG, the angles KAD, MBG are equal, as also the right angles AKD,

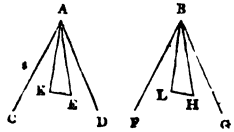


BMG, and that the sides AK, BM, adjacent to the equal angles, are equal to one another; therefore (I. 26) KD is equal to MG, and AD to BG. For the same reason, in the triangles KAL, MBN, KL is equal to MN, and AL to BN: and in the triangles LAD, NBG, LA, AD are equal to NB, BG, and (hyp.) they contain equal angles; therefore (I. 4) LD is equal to NG. Lastly, in the triangles KLD, MNG, the sides DK, KL are equal to GM, MN, and LD to NG; therefore (I. 8) the angle DKL is equal to GMN. But DKL is the inclination of the planes CAD, CAE, and GMN is the inclination of the planes FBG, FBH, which planes (XI. def. 2) have therefore the same inclination. In the same manner also, it may be demonstrated, that the other planes in which the equal angles are, have the same inclination to one another. Therefore, if two solid angles, &c.

Prop. B. Theor.—**Two solid angles, contained each by three plane angles which are equal to one another, each to each, and alike situated, are equal to one another.**

Let there be two solid angles at A and B, of which the solid angle at A is contained by the three plane angles CAD, CAE, EAD; and that at B, by the three plane angles FBG, FBH, HBG, which are equal to the other three, each to each: the solid angles at A and B are equal.

Let the solid angle at A be applied to the solid angle at B: and, first, the plane angle CAD being applied to FBG, so that the point A may coincide with B, and AC with BF; then AD coincides with BG, because the angle CAD is equal to FBG. And, because (XI. A) the inclination of the planes CAE, CAD is equal to the inclination of the planes FBH, FBG, the plane CAE coincides with FBH, because the planes CAD, FBG coincide with one another. Also, because AC, BF coincide, and the angle CAE is equal to FBH; therefore AE coincides with BH, and AD with BG; wherefore the plane EAD coincides with HBG. Therefore the solid angle A coincides with the solid angle B; and consequently (I. ax. 8) they are equal to one another. Two solid angles, therefore, &c.



Cor. If AE, BH be taken equal to each other; and EK be drawn perpendicular to the plane CAD, and HL to FBG; and if AK, BL be joined, EK is equal to HL, and the angle EAK to HBL. For, since the solid angles coincide, the points E, H coincide, because AE, BH are equal: also, since the planes CAD, FBG coincide, the perpendiculars EK, HL (XI. 13) coincide, and are equal: and since AE, AK coincide with BH, BL, the angles EAK, HBL are equal.*

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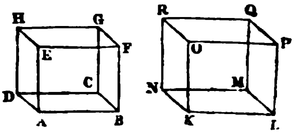
* This corollary is transferred to this proposition from the 35th of this book, for the purpose of facilitating the demonstration of the latter.

Prop. C. Theor.—Solid figures contained by the same number of equal and similar planes, alike situated, and having none of their solid angles contained by more than three plane angles, are equal and similar to one another.

Let AG, KQ be two solids contained by the same number of similar and equal planes, alike situated; viz. let the plane AC be similar and equal to KM; AF to KP; BG to LQ; GD to QN; DE to NO; and FH to PR; AG is equal and similar to KQ.

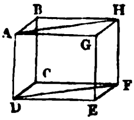
Because the plane angles BAD, BAE, EAD are (hyp.) equal to the plane angles LKN, LKO, OKN, each to each; therefore (XI. B) the solid angles at A and K are equal. In the same manner, the other solid angles of the figures are proved to be equal. If then AG be applied to KQ, first, the plane figure AC being applied to KM; AB coinciding with KL, AC must coincide with KM, because they are equal and similar: therefore the straight lines AD, DC, CB coincide with KN, NM, ML, each with each; and the points A, D, C, B, with K, N, M, L. Also the solid angles at A and K coincide: wherefore the plane AF coincides with KP, and the figure AF with KP, because they are equal and similar. Therefore the straight lines AE, EF, FB, coincide with KO, OP, PL; and the points E, F, with O, P. In the same manner, the figure AH coincides with KR, the straight line DH with NR, and the point H with R. Also because the solid angles at B and L are equal, it may be proved, in the same manner, that the figure BG coincides with LQ, the straight line CG with MQ, and the point G with Q. Since, therefore, all the planes and sides of AG coincide with those of KQ, AG is equal and similar (XI. def. 8) to KQ. And, in the same manner, any other solid figures whatever, contained by the same number of equal and similar planes, alike situated, and having none of their solid angles contained by more than three plane angles, may be proved to be equal and similar to one another. Therefore, solid figures, &c.



Prop. XXIV. Theor.—If a solid be contained by six planes, two and two of which are parallel, the opposite planes are similar and equal parallelograms.

Let the solid BE be contained by the parallel planes AC, GF: BG, CE; BF, AE; its opposite planes are similar and equal parallelograms.

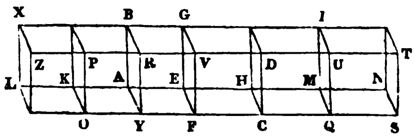
Because the parallel planes BG, CE are cut by the plane AC, their common sections AB, CD are (XI. 16) parallel. Also, because the parallel planes BF, AE are cut by the plane AC, their common sections AD, BC are parallel: and AB, CD are parallel; therefore AC is a parallelogram. In like manner, it may be proved, that CE, FG, GB, BF, AE are parallelograms. Join AH, DF; and because AB is parallel to DC, and BH to CF, the angle ABH is equal (XI. 10) to DCF. Then, because AB, BH are equal to DC, CF, and the angle ABH to DCF; the triangle ABH is equal (I. 4) to DCF. But (I. 34) the parallelogram BG is double of the triangle ABH, and CE is double of DCF; therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner, it may be proved, that AC is equal and similar to GF, and AE to BF. Therefore, if a solid, &c.



Prop. XXV. Theor.—A plane cutting a parallelepiped, and parallel to two of its opposite planes, divides the whole into two solids, the base of one of which is to the base of the other, as the one solid is to the other.

Let the parallelepiped ABCD be cut by the plane EV, which is parallel to the opposite planes AR, HD, and divides the whole into the two solids AV, ED; as the base AEFY of the first is to the base EHCF of the other, so is AV to ED.

Produce AH both ways, and take any number of straight lines HM, MN, each equal to EH, and any number AK, KL, each equal to EA, and complete the parallelograms LO, KY, HQ, MS, and the solids LP, KR, HU, MT. Then, because LK, KA, AE are all equal, the parallelograms LO, KY, AF (I. 36) are equal: and likewise the parallelograms KX, KB, AG; as also (XI. 24) the parallelograms LZ, KP, AR, because they are opposite planes. For the same reason, the parallelograms EC, HQ, MS, are equal; and the parallelograms HG, HI, IN, as also HD, MU, NT. Therefore three planes of the solid LP, are equal and similar to three planes of KR, as also to three planes of AV. But (XI. 24) the three planes opposite to these three are equal and similar to them in the several solids, and none of their solid angles are contained by more than three plane angles; therefore (XI. C) the solids LP, KR, AV are equal to one another. For the same reason, ED, HU, MT are equal to one another. Therefore whatever multiple the base LF is of AF, the same multiple is the solid LV of AV. For the same reason, whatever multiple the base NF is of HF, the same multiple is the solid TE of ED. And if the base LF be equal to NF, the solid LV is equal (XI. C) to ET; if greater, greater; and if less, less. Therefore (V. def. 5) as the base AF is to the base FH, so is the solid AV to the solid ED: wherefore a plane, &c.

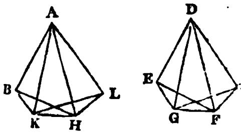


PROP. XXVI. PROB.—At a given point in a given straight line, to make a solid angle equal to a given solid angle contained by three plane angles.

Let A be a given point in a given straight line AB, and D a given solid angle contained by the three plane angles EDC, EDF, FDC; it is required to make at A in the straight line AB a solid angle equal to the solid angle D.

In DF take any point F, from which draw (XI. 11) FG perpendicular to the plane EDC, meeting that plane in G; join DG, and (I. 23) make the angle BAL equal to EDC, and in the plane BAL make the angle BAK equal to EDG; then make AK equal to DG, and (XI. 12) draw KH perpendicular to the plane BAL, and equal to GF, and join AH. Then the solid angle at A, which is contained by the plane angles, BAL, BAH, HAL, is equal to the given solid angle at D.

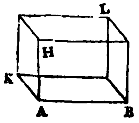
Take AB, DE equal to one another; and join HB, KB, FE, GE: and (XI. def. 1) because FG is perpendicular to the plane EDC, FGD, FGE are right angles. For the same reason, HKA, HKB are right angles: and because KA, AB are equal to GD, DE, each to each, and contain equal angles, BK is equal (I. 4) to EG: also KH is equal to GF, and HKB, FGE are right angles; therefore HB is equal to FE. Again, because AK, KH are equal to DG, GF, and contain right angles, AH is equal to DF; also AB is equal to DE, and HB to FE; therefore (I. 8) the angles BAH, EDF are equal. Again, since (const.) the angle BAL is equal to EDC, and BAK to EDG, the remaining angles KAL, GDC are (I. ax.3) equal to one another; and, by taking AL and DC equal, and joining LH, LK, CF, CG, it would be proved, as in the foregoing part, that the angle HAL is equal (I. 8) to FDC. Therefore, because the three plane angles BAL, BAH, HAL, which contain the solid angle at A, are equal to the three EDC, EDF, FDC, which contain the solid angle at D, each to each, and are situated in the same order, the solid angle at A is equal (XI. B) to the solid angle at D. Therefore what was required has been done.

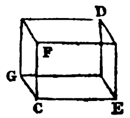


PROP. XXVII. PROB.—To describe from a given straight line a parallelepiped similar, and similarly situated to one given.

Let AB be the given straight line, and CD the given parallelepiped. It is required from AB to describe a parallelepiped similar and similarly situated to CD.

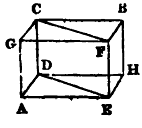
At the point A of the straight line AB, make (XI. 26) a solid angle equal to the solid angle at C, and let BAK, KAH, HAB be the plane angles which contain it, so that BAK is equal to ECG, KAH to GCF, and HAB to FCE; also as EC to CG, so make (VI. 12) BA to AK; and as GC to CF, so make KA to AH; wherefore, *ex æquo*, as EC : CF :: BA : AH. Complete the parallelogram BH, and the solid AL. Then, because EC : CG :: BA : AK, the sides about the equal angles ECG, BAK are proportionals; therefore the parallelogram BK is similar to EG. For the same reason, KH is similar to GF, and HB to FE. Therefore three parallelograms of the solid AL are similar to three of the solid CD; and (XI. 24) the three opposite ones in each are equal and similar to these, each to each. Also, because the plane angles which contain the solid angles of the figures are equal, each to each, and situated in the same order, the solid angles (XI B) are equal, each to each. Therefore (XI. def. 8) AL is similar to CD. Wherefore, from a given straight line AB a parallelepiped AL has been described similar, and similarly situated to the given one CD; which was to be done.





PROP. XXVIII. THEOR.—If a parallelepiped be cut by a plane passing through the diagonals of two of the opposite planes; it is bisected by that plane.

Let AB be a parallelepiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, viz. those which join the equal angles in each. Then (XI. 9) CD, FE are parallels, because each of them is parallel to GA; wherefore (XI. 7) the diagonals CF, DE are in the plane in which the parallels are, and (XI. 16) are themselves parallels. Again, because (I. 34) the triangle CGF is equal to CBF, and DAE to DHE; and that (XI. 24) the parallelogram CA is equal and similar to the opposite one BE; and GE to CH: therefore the prism contained by the two triangles CGF, DAE, and the three parallelograms CA, GE, EC, is equal (XI. C) to the prism contained by the two triangles CBF, DHE, and the three parallelograms BE, CH, EC; because they are contained by the same number of equal and similar planes, alike situated, and none of their solid angles are contained by more than three plane angles. Therefore, if a parallelepiped, &c.



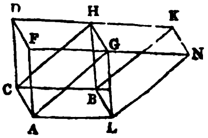
"N.B. The *insisting lines* of a parallelepiped, mentioned in some following propositions, are the sides of the parallelograms between the base and the opposite plane."

PROP. XXIX. THEOR.—Parallelepipeds upon the same base, and of the same altitude, the insisting lines of which are terminated in the same straight lines of the plane opposite to the base, are equal to one another.

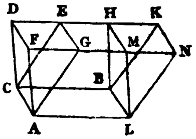
Let the parallelepipeds AH, AK (2d fig.) be upon the same base

AB, and of the same altitude; and let their insisting lines AF, AG, LM, LN, be terminated in the same straight line FN, and CD, CE, BH, BK in the same DK; AH is equal to AK.

First, let the parallelograms DG, HN, which are opposite to the base AB, have a common side HG Then because AH, is cut by the plane AGHC passing through the diagonals AG, CH, of the opposite planes ALGF, CBHD, AH is bisected (XI. 28) by the plane AGHC. For the same reason, AK is bisected by the plane LGHB through the diagonals LG, BH. Therefore the solids AH, AK are equal, each of them being double of the prism contained between the triangles ALG, CBH.



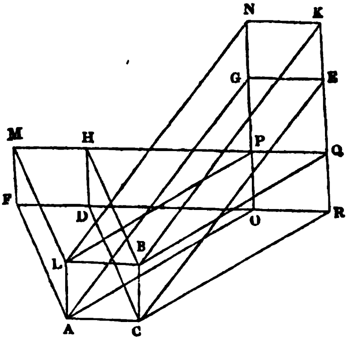
But let the parallelograms DM, EN, opposite to the base, have no common side. Then, because CH, CK are parallelograms, CB is equal (I. 34) to each of the opposite sides DH, EK; wherefore DH is equal to EK. From DK take separately, EK, DH; then DE is equal to HK: wherefore also (I. 38) the triangles CDE, BHK are equal: and (I. 36) the parallelogram DG is equal to HN. For the same reason, the triangle AFG is equal to LMN; and (XI. 24) the parallelogram CF is equal to BM, and CG to BN; for they are opposite. Therefore (XI. C) the prism which is contained by the two triangles AFG, CDE, and the three parallelograms AD, DG, GC, is equal to the prism contained by the two triangles LMN, BHK, and the three parallelograms BM, MK, KL. If therefore the prism LMNBHK be taken from the solid of which the base is the parallelogram AB, and in which FDKN is the one opposite to it; and if from the same solid there be taken the prism AFGCDE; the remaining parallelepipeds AH, AK are equal. Therefore, parallelepipeds, &c.



PROP. XXX. THEOR.—Parallelepipeds upon the same base and of the same altitude, the insisting lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the parallelepipeds CM, CN, be upon the same base AB, and of the same altitude, but their insisting lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the same straight lines: CM, CN are equal to one another.

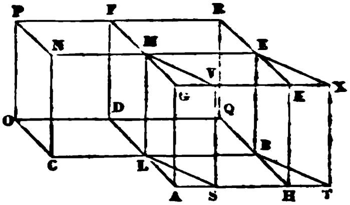
Produce FD, MH, and NG, KE, and let them meet one another in O, P, Q, R; and join AO, LP, BQ, CR. Then, because the plane LBHM is parallel to the opposite plane ACDF, and that LBHM is the plane in which are the parallels LB, MHPQ, in which also is the figure BLPQ; and ACDF is the plane in which are the parallels AC, FDOR, in which also is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes. In like manner, because the plane ALNG is parallel to the opposite plane CBKE, and that ALNG is the plane in which are the parallels AL, OPGN, in which also is the figure ALPO; and CBKE is the plane in which are the parallels CB, RQEK, in which also is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes; and the planes ACBL, ORQP are parallel; therefore the solid CP is a parallelepiped. But the solid CM, of which the base is ACBL, to which FDHM is the opposite parallelogram, is equal (XI. 29) to the solid CP, of which the base is the parallelogram ACBL, to which ORQP is the one opposite; because they are upon the same base, and their insisting straight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the same straight lines FR, MQ. For the same reason the solid CP is equal to CN; for they are upon the same base ACBL, and their insisting straight lines AO, AG, LP, LN; CR, CE, BQ, BK are in the same straight lines ON, RK; therefore (I. ax. 1) CM is equal to CN. Wherefore, parallelepipeds, &c.



Prob. XXXI. Theor.—Parallelepipeds which are upon equal bases, and of the same altitude, are equal to one another.

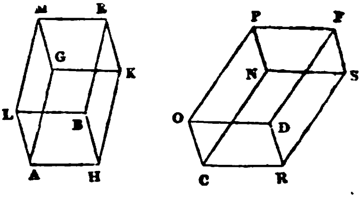
Let the parallelepipeds AE, CF, be upon equal bases AB, CD, and be of the same altitude; AE is equal to CF.

First, let the insisting lines be at right angles to the bases AB, CD, and let the bases be placed in the same plane, and so that the sides CL, LB may be in a straight line; therefore (XI. 13) the straight line LM, which is at right angles to the plane in which the bases are, in the point L, is common to the two solids AE, CF. Let the other insisting lines of the solids be AG, HK, BE; DF, OP, CN; and, first, let the angle ALB be equal to CLD; then (I. 14, cor.) AL, LD are in a straight line. Produce OD, HB, and let them meet in Q; complete also the parallelepiped LR, the base of which is the parallelogram LQ, and of which LM is one of its insisting lines. Therefore, (V. 7) because the parallelogram AB is equal to CD, as AB : LQ : : CD : LQ; and because the parallelepiped AR is cut by the plane LMEB, which is parallel to the opposite planes AK, DR; as the base AB is to the base LQ, so is (XI. 25) the solid AE to the solid LR. For the same reason, because the parallelepiped CR is cut by the plane LMFD, which is parallel to the opposite planes CP, BR; as the base CD to the base LQ, so is the solid CF to the solid LR. But AB : LQ :: CD : LQ, as has been proved; therefore (V. 11) as the solid AE to the solid LR, so is the solid CF to the solid LR; and therefore (V. 9) the solid AE is equal to the solid CF.



But let the parallelepipeds SE, CF be upon equal bases SB, CD, and be of the same altitude, and let their insisting lines be at right angles to the bases; and place the bases SB, CD in the same plane, so that CL, LB may be in a straight line; and let the angles SLB, CLD be unequal; the solid SE is also in this case equal to the solid CF. Produce DL, TS, till they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the solids AE, LR. Therefore (XI. 29) the solid AE, of which the base is the parallelogram LE, and AK the one opposite to it, is equal to the solid SE, of which the base is LE, and to which SX is opposite; for they are upon the same base LE, and of the same altitude, and their insisting lines, viz. LA, LS, BH, BT; MG, MV, EK, EX are in the same straight lines AT, GX. Also, because (I. 35) the parallelogram AB is equal to SB, for they are upon the same base LB, and between the same parallels LB, AT, and that the base SB is equal to CD; therefore the base AB is equal to the base CD, and the angle ALB to CLD. Therefore, by the first case, the solid AE is equal to the solid CF; but AE is equal to SE, as was demonstrated; therefore SE is equal to CF.

But, if the insisting straight lines AG, HK, BE; CN, &c. be not at right angles to the bases AB, CD; in this case likewise the solid AE is equal to the solid CF.

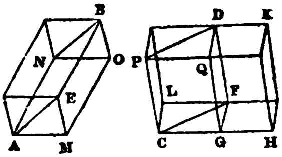


For if parallelepipeds be described on the bases AB, CD, of the same altitude as the solids AE, CF, and having their insisting lines perpendicular to their bases, they will be respectively equal (XI. 29, or 30) to AE, CF; and by the first case of this proposition they are equal to one another. Therefore (I. ax. 1) AE, CF are equal. Wherefore, parallelepipeds, &c.

PROP. XXXII. THEOR.—Parallelepipeds which have the same altitude, are to one another as their bases.

Let AB, CD be parallelepipeds of the same altitude; they are to one another as their bases; that is, as the base AE to the base CF, so is the solid AB to the solid CD.

To the straight line FG apply (I. 45, cor.) the parallelogram FH equal to AE, so that the angle FGH may be equal to the angle LCG; and complete the parallelepiped GK on the base FH, one of whose insisting lines is FD, the solids CD; GK being thus of the same altitude. Therefore (XI. 31) the solids AB, GK are equal, because they are upon equal bases AE, FH, and are of the same altitude. Now, the parallelepiped CK is cut by the plane DG, which is parallel to its opposite planes; therefore (XI. 25) the base HF is to GL, as the solid HD to DC. But HF is equal to AE, and the solid GK to AB; therefore, as the base AE to the base GL, so is the solid AB to the solid CD. Wherefore, parallelepipeds, &c.



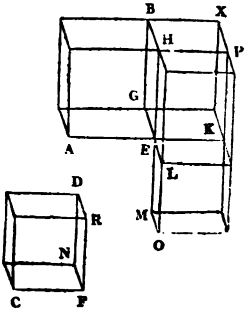
Cor. From this it is manifest that triangular prisms of the same altitude, are to one another as their bases.

Let the prisms, the bases of which are the triangles AEM, CFG, and NBO, PDQ the triangles opposite to them, have the same altitude; and complete the parallelograms AE, CF, and the parallelepipeds AB, CD, in the first of which let MO, and in the other let GQ be one of the insisting lines. And because the parallelepipeds AB, CD have the same altitude, they are to one another as the base AE is to GL; wherefore the prisms, which (XI. 28) are their halves, are to one another, as the base AE to GL; that is, (V. 15) as the triangle AEM to CFG.

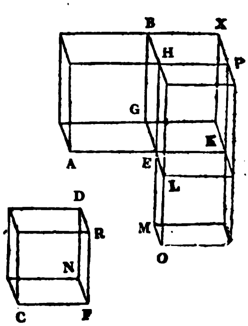
Prop. XXXIII. Theor.—Similar parallelepipeds are one to another in the triplicate ratio of their homologous sides.

Let AB, CD be similar parallelepipeds, and the side AE homologous to CF; AB has to CD the triplicate ratio of that which AE has to CF.

Produce AE, GE, HE, and in these produced take EK equal to CF, EL equal to FN, and EM equal to FR; and complete the parallelogram KL, and the solid KO. Because KE, EL are equal to CF, FN, and the angle KEL equal to the angle CFN, since it is equal to the angle AEG, which is equal to CFN, because the solids AB, CD are similar; therefore the parallelogram KL is similar and equal to CN. For the same reason, the parallelogram MK is similar and equal to CR, and also OE to FD. Therefore three parallelograms of the solid KO are equal and



similar to three parallelograms of the solid CD: and (XI. 24) the three opposite ones in each solid are equal and similar to these. Therefore (XI. C) the solid KO is equal and similar to CD. Complete the parallelogram GK, and the solids EX, LP upon the bases GK, KL, so that EH may be an insisting line in each of them; and thus they are of the same altitude with the solid AB. Then, because the solids AB, CD are similar (XI. def. 8, and alternately) as AE is to CF, so is EG to FN, and so is EH to FR; and FC is equal to EK, and FN to EL, and FR to EM: therefore, as AE to EK, so is EG to EL; and so is HE to EM. But (VI. 1) as AE to EK, so is the parallelogram AG to GK; and as GE to EL, so is GK to KL; and as HE to EM, so is BE to KM: therefore (V. 11) as the parallelogram AG to GK, so is GK to KL, and PE to KM. But (XI. 25) as AG to GK, so is the solid AB to EX; and as GK to KL, so is the solid EX to PL; and as PE to KM, so is the solid PL to KO: and therefore (V. 11) as the solid AB to EX, so is EX to PL, and PL to KO. But if four magnitudes be continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second: therefore the solid AB has to KO the triplicate ratio of that which AB has to EX. But as AB is to EX, so is the parallelogram AG to GK, and the straight line AE to EK. Wherefore the solid AB has to the solid KO, the triplicate ratio of that which AE has to EK; and the solid KO is equal to CD, and the straight line EK to CF. Therefore the solid AB has to CD the triplicate ratio of that which the side AE has to the homologous side CF. Wherefore, similar parallelepipeds, &c.



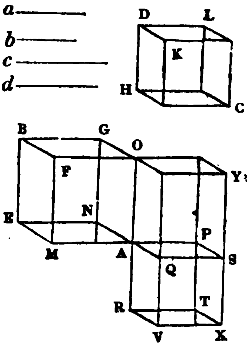
Cor. From this it is manifest, that, if four straight lines be continual proportionals, as the first is to the fourth, so is the parallelepiped described from the first to the similar solid similarly described from the second; because the first straight line has to the fourth the triplicate ratio of that which it has to the second.

PROP. D. THEOR.—Parallelepipeds contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

Let AB, CD be parallelepipeds, of which AB is contained by the parallelograms AE, AF, AG, equiangular, each to each, to CH, CK, CL, which contain CD. The ratio which AB has to

CD is the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH.

Produce MA, NA, OA to P, Q, R, making AP equal to DL, AQ to DK, and AR to DH; and complete the parallelepiped AX contained by the parallelograms AS, AT, AV, similar and equal to CH, CK, CL, each to each. Therefore (XI. C) AX is equal to CD. Complete likewise the solid AY, the base of which is AS, and of which AO is one of its insisting lines. Take any straight line a, and (VI. 12) find b, c, d, such that MA : AP :: a : b; NA : AQ :: b : c, and AO : AR :: c : d. Then, (VI. 23) because the parallelogram AE is equiangular to AS, AE is to AS, as a to c: and the solids AB, AY, being between the parallel planes BOY, EAS, are of the same altitude: therefore (XI. 32) the solid AB is to the solid AY, as AE to AS; that is, as a to c. Also (XI. 25) the solid AY is to AX as the base OQ is to QR; that is, (VI. 1) as OA to AR; that is, as c to d. And because the solid AB is to AY as a is to c, and AY to AX, as c is to d; *ex æquo*, AB is to AX or CD, which is equal to it, as a is to d. But (V. def. 10) the ratio of a to d is said to be compounded of the ratios of a to b, b to c, and c to d, which are the same with the ratios of the sides MA to AP, NA to AQ, and OA to AR, each to each: and the sides AP, AQ, AR are equal to the sides DL, DK, DH, each to each. Therefore, AB has to CD the ratio which is the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH. Parallelepipeds, therefore, &c.

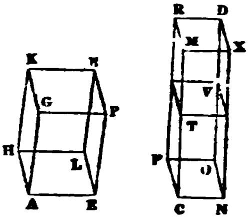


Prop. XXXIV. Theor.—The bases and altitudes of equal parallelepipeds are reciprocally proportional: and (2) if the bases and altitudes be reciprocally proportional, the parallelepipeds are equal.

1. Let AB, CD be equal parallelepipeds; their bases are reciprocally proportional to their altitudes; that is, as the base EH is to the base NP, so is the altitude of CD to the altitude of AB.

First, let the insisting lines AG, CM, &c. be at right angles to the bases. As the base EH to NP, so is CM to AG. If EH be equal to NP, then because AB is likewise equal to CD, CM is equal to AG. For, if EH, NP be equal, but AG, CM be not equal, neither is AB equal to CD. But (hyp.) the solids are equal. Therefore CM is not unequal to AG; that is, they are equal. Wherefore, as EH to NP, so is CM to AG.

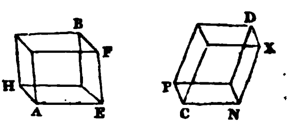
Next, let the bases EH, NP not be equal, but EH greater than the other. Since then the solid AB is equal to the solid CD, CM is greater than AG. For, if it be not, neither also, in this case, would the solids AB, CD be equal, which (hyp.) are equal. Make then CT equal to AG, and complete the parallelepiped CV, of which the base is NP and altitude CT. Because AB is equal to CD, therefore (V. 7) AB is to CV, as CD to CV. But (XI. 32) as AB to CV, so is EH to NP; for the solids AB, CV are of the same altitude; and (XI. 25) as CD to CV, so is the base MP to PT, and (VI. 1) so is the straight line MC to CT; and CT is equal to AG. Therefore, as EH to NP, so is MC to AG: wherefore (VI. def. 2) the bases of the parallelepipeds AB, CD are reciprocally proportional to their altitudes.



2. Let now the bases of the parallelepipeds AB, CD be reciprocally proportional to their altitudes: viz. as EH to NP, so let the altitude of CD be to the altitude of AB; AB is equal to CD. Let the insisting lines be, as before, at right angles to the bases. Then, if EH be equal to NP, since EH is to NP, as the altitude of CD is to the altitude of AB, therefore (V. A) the altitude of CD is equal to the altitude of AB. But (XI. 31) parallelepipeds upon equal bases, and of the same altitude, are equal to one another; therefore AB is equal to CD.

But let the bases EH, NP be unequal, and let EH be the greater. Therefore, since EH is to NP as CM, the altitude of CD, is to AG, the altitude of AB, CM is greater (V. A) than AG. Take CT equal to AG, and complete, as before, the solid CV. Then, because EH is to NP, as CM to AG, and that AG is equal to CT, therefore EH is to NP, as MC to CT. But as EH is to NP, so is AB to CV: for AB, CV are of the same altitude; and as MC to CT, so (VI. 1) is the base MP to PT, and (XI. 25) the solid CD to CV; and therefore as AB to CV, so is CD to CV; wherefore (V. 9) AB is equal to CD.

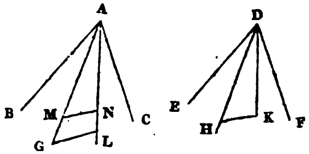
Second general case. 1. Let the insisting lines FE, XN, &c. not be at right angles to the bases of the solids. Then, if a parallelepiped be described on the base EH having the same altitude as AB, and having its insisting lines perpendicular to its base, it will (XI. 29, or 30) be equal to AB. For the same reason, if a parallelepiped be described on the base NP of the same altitude as CD, and having its insisting lines perpendicular to its base, it will be equal to CD. These parallelepipeds, therefore, are equal, if AB, CD be equal: and, by the first general case, their bases and altitudes, which are also the bases and altitudes of AB, CD, are reciprocally proportional.



2. Next, let the bases EH, NP be reciprocally proportional to the altitudes; AB is equal to CD. For, if the same construction were made, the bases and altitudes of the parallelepipeds so constructed would be reciprocally proportional to one another, being the same as those of AB, CD: and, therefore, by the first general case, those parallelepipeds are equal; and so also are AB, CD, which are equal to them. Therefore the bases, &c.

Prop. XXXV. Theor.—If, from the vertices of two equal plane angles, there be drawn two straight lines elevated above the planes in which the angles are, and containing equal angles with the sides of those angles, each to each; and if from any points in the lines above the planes perpendiculars be drawn to those planes; and if from the points in which they meet the planes, straight lines be drawn to the vertices of the first-mentioned angles; these straight lines make equal angles with the straight lines above the planes.

Let BAC, EDF be two equal plane angles; and from A, D let the straight lines AG, DH be elevated above the planes of the angles, making the angle GAB equal to HDE, and GAC to HDF; from any points G, H, in those lines let perpendiculars GL, HK be drawn (XI. 11) to the planes BAC, EDF meeting those planes in L, K, and join LA, KD: the angles GAL, HDK, are equal.



If AG, DH be equal, then the angle GAL is equal to HDK, as was shown in the corollary to proposition B of this book.

But if AG, DH be unequal, make AM equal to DH, and (I. 31) draw MN parallel to GL, meeting AL, the common section of the planes BAC, GAL, in N. Then (XI. 8) MN is perpendicular to the plane BAC, because GL, which is parallel to MN, is perpendicular to BAC; and, therefore, by the first case, the angle MAN, that is, GAL, is equal to HDK.* If, therefore, &c.

Prop. XXXVI. Theor.—If three straight lines be proportionals, any parallelepiped described from all three as its sides, is equal to the equilateral parallelepiped described from the mean proportional, one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other figure.

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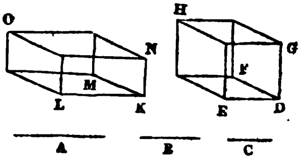
[*] This short and easy demonstration is substituted for the very tedious one given by Euclid.

Let A : B :: B : C; any parallelepiped described from A, B, C, is equal to the equilateral parallelepiped described from B, equiangular to the other.

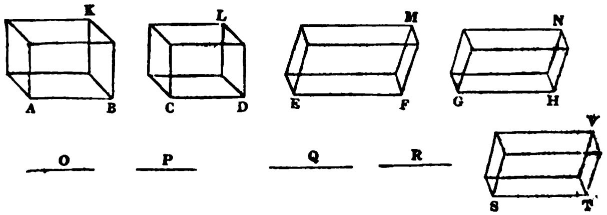


Take a solid angle D contained by three plane angles EDF, FDG, GDE; and make ED, DF, DG, each equal to B, and complete the parallelepiped DH. Make LK equal to A, and (XI. 26) at the point K in the straight line LK make a solid angle contained by the three plane angles LKM, MKN, NKL equal to the angles EDF, FDG, GDE, each to each; make also KN equal to B, and KM equal to C; and complete the parallelepiped KO. Then, because A : B :: B : C, and that A is equal to LK, and B to each of the lines DE, DF, and C to KM; therefore LK : ED :: DF : KM; that is, the sides about the equal angles are reciprocally proportional; therefore (VI. 14) the parallelogram LM is equal to EF. Also, because EDF, LKM are two equal plane angles, and the two equal straight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their sides; therefore (XI. B, cor.) the perpendiculars from G, N, to the planes EDF, LKM, are equal. The solids KO, DH, therefore, are of the same altitude; and since they are upon equal bases LM, EF, they are equal (XI. 31) to one another. If, therefore, &c.

PROP. XXXVII. THEOR.—If four straight lines be proportionals, the similar parallelepipeds similarly described from them are also proportionals: and (2) if the similar parallelepipeds similarly described from four straight lines be proportionals, the lines are proportionals.

1. Let the four straight lines AB, CD, EF, GH be proportionals, viz. as AB to CD, so EF to GH; and let the similar parallelepipeds AK, CL, EM, GN be similarly described from them: AK is to CL, as EM to GN.

Make (VI. 11) AB, CD, O, P continual proportionals, as also



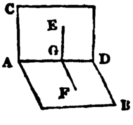
EF, GH, Q, R. Then, because AB : CD :: EF : GH: and that (V. 11) CD : O :: GH : Q, and O : P :: Q : R; therefore, *ex* *æquo*, AB : P :: EF : R. But (XI. 33, cor.) as AB to P, so is AK to CL; and as EF to R, so is EM to GN: therefore (V. 11) as AK to CL, so is EM to GN.

2. Next, let AK be to CL, as EM to GN: the straight line AB is to CD as EF to GH.

Take (VI. 12) AB to CD, as EF to ST, and (XI. 27) from ST describe a parallelepiped SV similar and similarly situated to either EM or GN. Then, by the preceding part, AK : CL :: EM : SV. But (hyp.) AK : CL :: EM : GN: therefore (V. 9) GN is equal to SV: it is likewise similar and similarly situated to SV; therefore the planes which contain the solids GN, SV are similar and equal, and their homologous sides GH, ST equal to one another: and because AB : CD :: EF : ST, and that ST is equal to GH; AB : CD :: EF : GH. If, therefore, &c.

PROP. XXXVIII. THEOR.—If one plane be perpendicular to another, a straight line drawn from a point in one of them perpendicular to the other, falls on their common section.

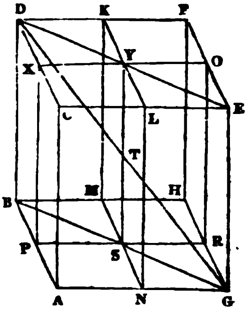
Let the plane CD be perpendicular to AB, and let AD be their common section; if any point E be taken in CD, the perpendicular drawn from E to AB will fall on AD.



From E draw EG in the plane CD perpendicular to AD, and in AB draw GF also perpendicular to AD. Then (XI. def. 3) EG is perpendicular to GF; and (const.) it is perpendicular to AD: it is therefore (XI. 4) perpendicular to the plane AB: and (XI. 13) through E no other straight line can be drawn perpendicular to AB. Therefore the perpendicular from E to AB falls on AD. Wherefore, if one plane, &c.

PROP. XXXIX. THEOR.—In a parallelepiped, if the sides of two of the opposite planes be each bisected, the common section of the planes passing through the points of division, and any diagonal of the parallelepiped bisect each other.

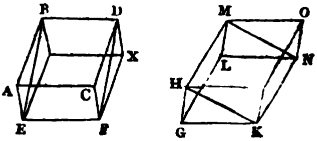
Let the sides of the opposite planes CF, AH of the parallelepiped AF, be bisected in the points K, L, M, N; X, O, P, R; and join KL, MN, XO, PR. Then (I. 33) because DK, CL are equal and parallel, KL is parallel to DC. For the same reason, MN is parallel to BA. But BA is parallel to DC; therefore (XI. 9) KL is parallel to BA. Also (XI. 9) because KL, MN are each of them parallel to BA, KL is parallel to MN; wherefore (I. def. 11) KL, MN are in one plane. In like manner, it may be proved, that XO, PR are in one plane. Let YS be the common section of the planes KN, XR; and DG a diagonal of the parallelepiped AF: YS and DG meet and bisect one another.



Join DY, YE, BS, SG. Because DX is equal to OE, and XY to YO, and (I. 29) the contained angles DXY, YOE are equal; DY is equal (I. 4) to YE, and the angle XYD to OYE; and therefore DYE (I. 14, cor.) is a straight line. For the same reason, BSG is a straight line, and BS equal to SG. Again, because CA is equal and parallel to DB, and also to EG; therefore (I. ax. 1, and XI. 9) DB is equal and parallel to EG; and DE, BG join their extremities; therefore (I. 33) DE is equal and parallel to BG; also DG, YS are drawn from points in the one to points in the other; they are therefore in one plane, and must meet one another: let them meet in T. Then, because DE is parallel to BG, the alternate angles EDT, BGT are (I. 29) equal; also (I. 15) the angles DTY, GTS are equal; and therefore in the triangles DTY, GTS, since the sides DY, GS are equal, being halves of DE, BG: DT is equal (I. 26) to TG, and YT equal to TS. Wherefore, in a parallelepiped, &c.

Prop. XL. Theor.—If there be two triangular prisms of the same altitude, the base of one of which is a parallelogram, and that of the other a triangle; if the parallelogram be double of the triangle, the prisms are equal.

Let the prisms ABCDEF, GHKLMN be of the same altitude, the first of which is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN and the three parallelograms LH, HN, NG; and let the parallelogram AF be taken as the base of one of them, and the triangle GHK, as base of the other; if AF be double of GHK, the prisms are equal.



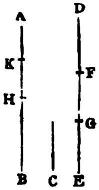
Complete the parallelepipeds AX, GO; and because the parallelogram AF is double of the triangle GHK; and (I. 34) the parallelogram HK double of the same triangle; therefore the bases AF, HK are equal. But (XI. 31) parallelepipeds upon equal bases, and of the same altitude are equal to one another. Therefore AX is equal to GO; and (XI. 28) the prism ABCDEF is half of AX; and the other prism half of GO. Therefore (I. ax. 7) the prisms are equal. Wherefore, &c.

BOOK XII.

Lemma I.*—**If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and so on: there will at length remain a magnitude less than the least of the proposed magnitudes.**

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken more than its half, and from the remainder more than its half, and so on; there will at length remain a magnitude less than C.

For C may be multiplied so as at length to become greater than AB. Let it be so multiplied, till DE, its multiple, is greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and so on, until there be as many divisions in AB as there are in DE: and let the divisions in AB be AK, KH, HB; and the divisions in ED be DF, FG, GE. Then, because DE is greater than AB, and that EG taken from DE is not greater than its half; but BH taken from AB is greater than its half; therefore the remainder GD is greater than the remainder HA. Again, because GD is greater than HA, and that GF is not greater than the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK. And FD is equal to C, therefore AK is less than C.



And, if only the halves be taken away, the same thing may in the same way be demonstrated. If, therefore, &c.

PROPOSITIONS.

Prop. I. Theor.—Similar polygons inscribed in circles, are to one another as the squares of the diameters.

In the circles ACE, FHL, let there be the similar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles: the square of BM is to the square of GN as the polygon AC is to the polygon FH.

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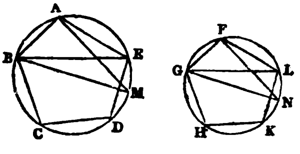
* This, which is the first proposition of the tenth book, is necessary for the demonstration of some of the propositions of this book.

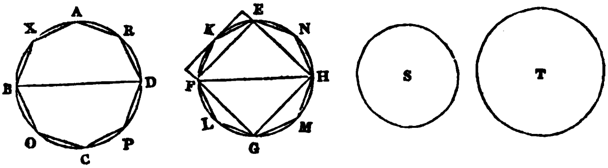
Join BE, AM, GL, FN. Then, because the polygons are similar, and (VI. 20) similar polygons are divided into similar triangles; the triangles ABE, FGL are similar, and (VI. def. 1) equiangular; the angle AEB therefore is equal to FLG. But (III. 21) AEB is equal to AMB, because they stand on the same arc; and for the same reason, the angles FLG, FNG are equal: therefore also the angles AMB, FNG are equal: and the right angles (III. 31) BAM, GFN are equal: therefore (VI. 4, schol. and alternately) BM : GN :: BA : GF; and therefore (*ex æquo*, and V. def. 11) the duplicate ratio of BM to GN, is the same with the duplicate ratio of BA to GF. But (VI. 20) the ratio of the square of BM to the square of GN, is the duplicate ratio of that which BM has to GN; and the ratio of the polygon AC to the polygon FH, is the duplicate of that which BA has to GF. Therefore (V. 11) the square of BM is to the square of GN, as the polygon AC to the polygon FH. Wherefore, similar polygons, &c.



PROP. II. THEOR.—Circles are to one another as the squares of their diameters.

Let AC, EG be two circles, and BD, FH their diameters: as the square of BD to the square of FH, so is AC to EG.

For if it be not so, the square of BD will be to the square of FH, as the circle AC is to some space either less than the circle EG, or greater than it. First, let it be to a space S less than EG; and in EG describe the square EFGH. This square is greater than half the circle EG; because, if through the points E, F, G, H, there be drawn tangents to the circle, the square EFGH is half (I. 41) of the square described about the circle; and the circle is less than the square described about it; therefore the square EFGH is greater than half the circle. Bisect the arcs EF, FG, GH, HE, in the points K, L, M, N, and join EK, KF, &c. Then each of the triangles EKF, FLG, &c. is greater than half of the segment of the circle in which it stands; because, if straight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the straight lines EF, FG, &c. be completed, each of the triangles EKF, FLG, &c. is half (I. 41) of the parallelogram in which it is: but every segment is less than the parallelogram in which it is: wherefore each of the triangles EKF, FLG, &c. is greater than half the segment of the circle which contains it. And if each of the arcs EK, KF, &c. be bisected, and their extremities be joined by straight lines, by continuing to do this, there will at length remain segments of the circle which, by the preceding lemma, will be together less than the excess of the circle EFGH above the space S. Let then the segments EK, KF, FL, LG, GM, MH, HN, NE be those that remain and are together less than the excess of the circle EFGH above S; then the rest of the circle, viz. the polygon EKFLGMHN, is greater than S. Describe likewise in the circle AC the polygon AXBOCPDR similar to the polygon EKFLGMHN: then (XII. 1) as the square of BD is to the square of FH, so is the first of these polygons to the second.

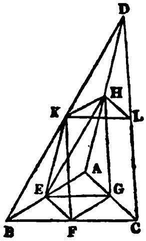


But the square of BD is also to the square of FH, as the circle AC is to S: therefore (V. 11) as the circle AC is to S, so is the polygon AXBOCPDR to the polygon EKFLGMHN. But the circle AC is greater than the polygon contained in it; wherefore (V. 14) the space S is greater than the polygon EKFLGMHN: but it is likewise less, as has been demonstrated; which is impossible. Therefore the square of BD is not to the square of FH, as the circle AC is to any space less than the circle EG. In the same manner, it may be demonstrated, that neither is the square of FH to the square of BD, as the circle EG is to any space less than the circle AC. Nor is the square of BD to the square of FH, as the circle AC is to any space greater than the circle EG. For, if possible, let it be so to T, a space greater than the circle EG: therefore inversely as the square of FH to the square of BD, so is the space T to the circle AC. But as T is to the circle AC, so is the circle EG to some space, which (V. 14) must be less than the circle AC, because T is greater, by hypothesis, than the circle EG. Therefore as the square of FH is to the square of BD, so is the circle EG to a space less than the circle AC, which has been demonstrated to be impossible. Therefore the square of BD is not to the square of FH, as the circle AC is to any space greater than the circle EG; and it has been demonstrated, that neither is the square of BD to the square of FH, as the circle AC to any space less than the circle EG: wherefore, as the square of BD to the square of FH, so is the circle AC to the circle EG. Circles, therefore, &c.

PROP. III. THEOR.—A triangular pyramid may be divided into two equal and similar triangular pyramids, which are similar to the whole pyramid: and into two equal prisms, which together are greater than half of the whole pyramid.

Let there be a pyramid, of which the base is the triangle ABC, and the vertex D: the pyramid ABCD may be divided into two equal and similar triangular pyramids, which are similar to the whole; and into two equal prisms which together are greater than half of the whole pyramid.

Bisect AB, BC, CA, AD, DB, DC, in the points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Because AE is equal to EB, and AH to HD, HE is parallel (VI. 2) to DB. For the same reason, HK is parallel to AB; and therefore HEBK is a parallelogram, and (I. 34) HK is equal to EB. But EB is equal to AE; therefore also AE is equal to HK; and AH is equal to HD; also (I. 29) the angle EAH is equal to KHD; therefore (I. 4) in the triangles HAE, DHK, EH is equal to KD, and the triangle AEH is equal and similar to the triangle HKD. For the same reason, the triangle AGH is equal and similar to HLD: and (XI. 10) because the two straight lines EH, HG are parallel to KD, DL, the angles EHG, KDL are equal: also EH, HG are equal to KD, DL, each to each; and therefore (I. 4) EG is equal to KL, and the triangle EHG equal and similar to KDL. For the same reason, the triangle AEG is also equal and similar to HKL. Therefore (XI. C) the pyramid AEGH, of which the base is the triangle AEG, and the vertex H, is equal and similar to the pyramid HKLD, the base of which is the triangle KHL, and the vertex D. Also, because HK is parallel to AB, a side of the triangle ADB, the triangle ADB is equiangular to the triangle HDK, and (VI. 4) their sides are proportionals: they are therefore similar. For the same reason, the triangles DBC, DKL are similar: so also are the triangles ADC, HDL; and ABC, AEG. But the triangles AEG, HKL have been proved to be similar: therefore (VI. 21) the triangle ABC is similar to HKL. The pyramid ABCD is therefore similar (XI. B, and def. 8) to the pyramid HKLD. But the pyramid HKLD is similar, as has been proved, to the pyramid AEGH; wherefore the pyramid ABCD is similar to the pyramid AEGH. Therefore each of the pyramids AEGH, HKLD is similar to the whole pyramid ABCD. Again, because BF is equal to FC, the parallelogram EBFG is double (I. 41) of the triangle GFC. But (XI. 40) when there are two prisms of the same altitude, of which one has a parallelogram for its base, and the other a triangle that is half of the parallelogram, these prisms are equal; therefore the prism having the parallelogram EBFG for its base, and the straight line KH opposite to it, is equal to the prism having the triangles GFC, HKL for its bases; for they are of the same altitude, because they are between the parallel planes (XI. 15) ABC, HKL. It is also manifest, that each of these prisms is greater than either of the pyramids AEGH, HKLD, because, if EF be joined, the prism having the parallelogram EBFG for its base, and KH the line opposite to it, is greater than the pyramid EBFK; but this pyramid (XI. C) is equal to the pyramid AEGH, because they are contained by equal and similar planes. Therefore the

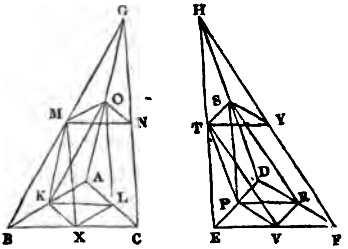


prism having EBFG for its base, and opposite side KH, is greater than the pyramid AEGH: and the prism of which the base is EBFG, and opposite side KH, is equal to the prism having for its bases the triangles GFC, HKL: and the pyramids AEGH, HKLD are equal; therefore the two prisms before-mentioned are greater than these two pyramids. Therefore the whole pyramid ABCD is divided into two equal pyramids similar to one another, and to the whole; and into two equal prisms; and the two prisms are together greater than half of the whole pyramid. A triangular pyramid, therefore, &c.

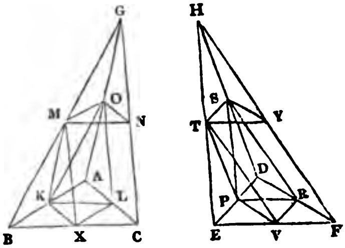
Prop. IV. Theor.—If two triangular pyramids of the same altitude be each divided into two equal pyramids, similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on: as the base of one of the first two pyramids is to the base of the other, so are all the prisms in one of them to all the prisms in the other, that are produced by the same number of divisions.

Let there be two pyramids of the same altitude upon the triangular bases ABC, DEF, and having their vertices in the points G, H; and let each of them be divided into two equal pyramids similar to the whole, and into two equal prisms; and let each of the pyramids thus made be conceived to be divided in like manner, and so on: as ABC is to DEF, so are all the prisms in the pyramid ABCG to all the prisms in the pyramid DEFH, made by the same number of divisions.

Make the same construction as in the foregoing proposition: and because BX, XC are equal, and also AL, LC, therefore (VI. 2) XL is parallel to AB, and the triangle ABC is similar to LXC. For the same reason, DEF is similar to RVF. Now, because BC is double of CX, and EF of FV, therefore BC : CX :: EF : FV; and upon BC, CX are described the similar and similarly situated rectilineal figures ABC, LXC; and upon EF, FV, in like manner, are described the similar figures DEF, RVF: therefore (VI. 22) as the triangle ABC is to LXC, so is the triangle DEF to RVF, and, alternately, as the triangle ABC to DEF, so is the triangle LXC to RVF. Again, because (XI. 15) the planes ABC, OMN, as also DEF, STY are parallel, the perpendiculars drawn from G, H to the bases ABC, DEF, which (hyp.) are equal to one another, will



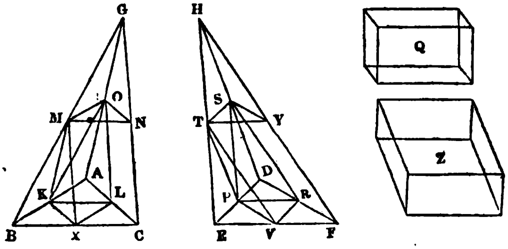
be each bisected (XI. 17) by the planes OMN, STY, because the straight lines GC, HF are bisected in the points N, Y, by the same planes. Therefore the prisms LXCOMN, RVFSTY are of the same altitude; and therefore (XI. 32, cor.) as the base LXC to the base RVF, that is, (V. 11) as the triangle ABC to DEF, so is the prism having the triangles LXC, OMN for its bases, to the prism of which the bases are the triangles RVF, STY. And because the two prisms in the pyramid ABCG are equal to one another, as also the two in the pyramid DEFH; as the prism of which the base is the parallelogram KBXL, and opposite side MO to the prism having the triangles LXC, OMN for its bases; so (V. 7) is the prism of which the base is the parallelogram PEVR, and opposite side TS, to the prism of which the bases are the triangles RVF, STY. Therefore, by composition, as the prisms KBXLMO, LXCOMN together are to the prism LXCOMN; so are the prisms PEVRTS, RVFSTY to the prism RVFSTY: and, alternately, as the prisms KBXLMO, LXCOMN are to the prisms PEVRTS, RVFSTY; so is the prism LXCOMN, to the prism RVFSTY. But as the prism LXCOMN to the prism RVFSTY, so is, as has been proved, ABC to DEF: therefore (V. 11) as ABC to DEF, so are the two prisms in the pyramid ABCG to the two in the pyramid DEFH. In like manner, if the pyramids now made, for example, the two OMNG, STYH, be divided similarly; as the base OMN is to the base STY, so will the two prisms in the pyramid OMNG be to the two in the pyramid STYH. But OMN : STY :: ABC : DEF; therefore (V. 11) as ABC to DEF, so are the two prisms in the pyramid ABCG to the two in the pyramid DEFH; and so are the two prisms in the pyramid OMNG to the two in the pyramid STYH; and so are all four to all four: and the same may be shown of the prisms made by dividing the pyramids AKLO and DPRS, and of all made by the same number of divisions. Therefore, if two triangular pyramids, &c.



Prop. V. Theor.—Triangular pyramids, of the same altitude, are to one another as their bases.

Let the pyramids ABCG, DEFH, of which the vertices are G, H, be of the same altitude: as the base ABC to the base DEF, so is the pyramid ABCG to the pyramid DEFH.

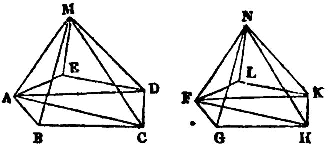
For, if it be not so, ABC must be to DEF, as the pyramid ABCG to a solid either less than the pyramid DEFH, or greater than it. First, let it be to a solid Q less than it: and divide DEFH into two equal pyramids similar to the whole, and into two equal prisms: these two prisms (XII. 3) are greater than the half of the whole pyramid. Again, let the pyramids made by this division be in like manner divided, and so on, until the pyramids which remain undivided in DEFH, are all of them together, less than the excess of DEFH above Q; which (XII. lem. 1) it is possible to do. Let these, for example, be the pyramids DPRS, STYH. Therefore the prisms, which make the rest of the pyramid DEFH, are greater than Q. Divide likewise the pyramid ABCG in the same manner, and into as many parts as DEFH. Therefore (XII. 4) as the base ABC is to the base DEF, so are the prisms in ABCG to those in DEFH. But as ABC to DEF, so, by hypothesis, is the pyramid ABCG to Q; and therefore, as ABCG is to Q, so are the prisms in ABCG to those in DEFG. But ABCG is greater than the prisms contained in it; wherefore also (V. 14) Q is greater than the prisms in DEFH. But it is also less, which is impossible. Therefore ABC is not to DEF, as ABCG to any solid which is less than DEFH. In the same manner, it may be demonstrated, that DEF is not to ABC, as DEFH to any solid which is less than ABCG. Nor can ABC be to DEF, as ABCG to any solid which is greater than DEFH. For, if it be possible, let it be so to a greater, Z. Then, because ABC is to DEF, as ABCG to Z; by inversion, as DEF to ABC, so is Z to ABCG. But as Z is to ABCG, so is DEFH to some solid, which (V. 14) must be less than ABCG, because Z is greater than DEFH. And, therefore, as DEF to ABC, so is DEFH to a solid less than ABCG; the contrary to which has been proved. Therefore ABC is not to DEF, as ABCG to any solid which is greater than DEFH. And it has been proved, that neither is ABC to DEF, as ABCG to any solid which is less than DEFH. Therefore, as ABC is to DEF, so is ABCG to DEFH. Wherefore, &c.



Prop. VI. Theor.—Pyramids of the same altitude which have polygons for their bases, are to one another as their bases.

Let the pyramids which have the polygons ABCDE, FGHKL for their bases, and their vertices in the points M, N, be of the same altitude: as the base ABCDE to the base FGHKL, so is the pyramid ABCDEM to the pyramid FGHKLN.

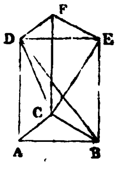
Divide the base ABCDE into the triangles ABC, ACD, ADE; and the other base into the triangles FGH, FHK, FKL; and upon the bases ABC, ACD, ADE let there be as many pyramids having the common vertex M, and upon the remaining bases as many pyramids having the common vertex N. Then, since (XII. 5) the triangle ABC is to FGH, as the pyramid ABCM to FGHN; and the triangle ACD to FGH, as the pyramid ACDM to FGHN; and also the triangle ADE to FGH, as the pyramid ADEM to FGHN; as all the first antecedents to their common consequent, so (V. 24, cor. 2) are all the other antecedents to their common consequent; that is, as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHN; and, for the same reason, as the base FGHKL to the base FGH, so is the pyramid FGHKLN, to the pyramid FGHN; and, by inversion, as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid FGHKLN. Then, because the base ABCDE is to the base FGH, as the pyramid ABCDEM to the pyramid FGHN; and the base FGH to the base FGHKL, as the pyramid FGHN to the pyramid FGHKLN; therefore, *ex æquo*, as the base ABCDE to the base FGHKL, so is the pyramid ABCDEM to the pyramid FGHKLN. Therefore, &c.



PROP. VII. THEOR.—Every triangular prism may be divided into three triangular pyramids, equal to one another.

Let there be a prism of which the bases are the triangles ABC, DEF: the prism may be divided into three equal triangular pyramids.

Join BD, EC, CD; and because ABED is a parallelogram, and BD its diagonal, the triangles ABD, EBD are (I. 34) equal; therefore (XII. 5) the pyramid of which the base is ABD, and vertex C, is equal to the pyramid of which the base is EBD and vertex C. But EBC may be taken as the base of this pyramid, and D as its vertex. It is therefore equal (XII. 5) to the pyramid of which ECF is the base, and D the vertex; for they have the same altitude, and (I. 34) equal bases ECF, ECB; and it has been already proved to be equal to the pyramid ABDC. Therefore the prism ABCDEF is divided into three equal pyramids having triangular bases, viz. into the pyramids ABDC, EBDC, ECFD. Therefore, every triangular prism, &c.



Cor. 1. From this it is manifest, that every pyramid is the third part of a prism which has the same base, and an equal altitude;

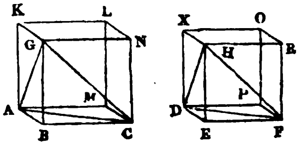
for if the base of the prism be any other rectilineal figure than a triangle, it may be divided into prisms having triangular bases.

Cor. 2. Prisms of equal altitudes are to one another as their bases; because the pyramids upon the same bases, and of the same altitude, are (XII. 6) to one another as their bases.

PROP. VIII. THEOR.—Similar triangular pyramids are one to another in the triplicate ratio of their homologous sides.

Let the pyramids of which ABC, DEF are the bases, and G, H the vertices, be similar, and similarly situated; the pyramid ABCG has to DEFH, the triplicate ratio of that which the side BC has to the homologous side EF.

Complete the parallelograms BM, BN, BK, and the parallelepiped BGML contained by these planes and those opposite to them: and, in like manner, complete the parallelepiped EHPO contained by the three parallelograms EP, ER, EX, and those opposite to them. Then, because the pyramids ABCG, DEFH are similar, the angle ABC is equal (XI. def. 8) to DEF, GBC to HEF, and ABG to DEH: and (VI. def. 1) AB : BC : : DE : EF; that is, the sides about the equal angles of the parallelograms BM, EP are proportionals: wherefore BM is similar to EP. For the same reason, BN is similar to ER, and BK to EX. Therefore the three parallelograms BM, BN, BK are similar to the three EP, ER, EX: and (XI. 24) the three BM, BN, BK are equal and similar to the three which are opposite to them, and the three EP, ER, EX equal and similar to the three opposite to them: wherefore the solids BGML, EHPO are contained by the same number of similar planes; and (XI. B) their solid angles are equal; the solid BGML is therefore similar (XI. def. 8) to EHPO. But (XI. 33) similar parallelepipeds have the triplicate ratio of that which their homologous sides have: therefore BGML has to EHPO the triplicate ratio of that which BC has to the homologous side EF. But (V. 15) as BGML is to EHPO, so is the pyramid ABCG to the pyramid DEFH; because the pyramids are the sixth parts of the solids, since the prism, which (XI. 28) is half of the parallelepiped, is triple (XII. 7) of the pyramid. Therefore also the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which BC has to the homologous side EF. Therefore, &c.

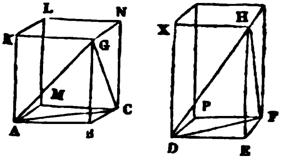


Cor. From this it is evident that similar pyramids which have bases of more sides than three, are likewise to one another in the triplicate ratio of their homologous sides. For they may be divided into similar triangular pyramids, because the similar figures which are their bases, may be divided into the same number of similar triangles, homologous to the whole bases; therefore as one of the triangular pyramids in the first pyramid is to one of the triangular pyramids in the other, so are all the triangular pyramids in the first to all in the other; that is, so is the first whole pyramid to the other. But one triangular pyramid is to a similar triangular pyramid, in the triplicate ratio of their homologous sides; and therefore the first whole pyramid has to the other, the triplicate ratio of that which a side of the first has to the homologous side of the other.

Prop. IX. Theor.—The bases and altitudes of equal triangular pyramids are reciprocally proportional: and (2) triangular pyramids having their bases and altitudes reciprocally proportional, are equal to one another.

1. Let the pyramids of which the triangles ABC, DEF are the bases, and G, H, the vertices, be equal: their bases and altitudes are reciprocally proportional, viz. ABC is to DEF, as the altitude of the pyramid DEFH to the altitude of ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, HF; and the parallelepipeds BGLM, EHOP contained by these planes and those opposite to them. Then, because the pyramid ABCG is equal to DEFH, and that (I. 34, and XII. 7, cor. 1) the solid BGLM is sextuple of ABCG, and EHOP of DEFH; therefore (V. ax. 1) BGLM is equal to EHOP. But (XI. 34) the bases and altitudes of equal parallelepipeds are reciprocally proportional; therefore as BM to EP, so is the altitude of EHPO to the altitude of BGML. But (V. 15) BM is to EP as the triangle ABC to DEF; therefore as ABC to DEF, so is the altitude of EHOP to the altitude of BGLM. But the altitude of EHOP is the same with the altitude of the pyramid DEFH; and the altitude of BGLM is the same with the altitude of the pyramid ABCG: therefore, as ABC to DEF, so is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: wherefore the bases and altitudes of the pyramids ABCG, DEFH are reciprocally proportional.



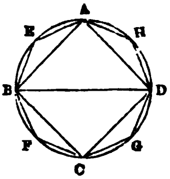
2. Again, let the bases and altitudes of the pyramids ABCG, DEFH be reciprocally proportional, viz. ABC to DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: ABCG is equal to DEFH.

The same construction being made, because as ABC to DEF, so is the altitude of the pyramid DEFH to the altitude of ABCG: and as ABC to DEF, so is BM to EP; therefore BM is to EP, as the altitude of the pyramid DEFH to the altitude of ABCG. But the altitude of DEFH is the same with the altitude of the parallelepiped EHOP; and the altitude of ABCG is the same with the altitude of the parallelepiped BGLM. As, therefore, the base BM to the base EP, so is the altitude of the parallelepiped

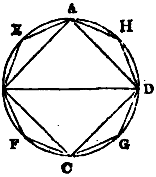
EHOP to the altitude of the parallelepiped BGLM. But (XI. 34) parallelepipeds having their bases and altitudes reciprocally proportional, are equal. Therefore the parallelepiped BGLM is equal to the parallelepiped EHOP. And the pyramid ABCG is the sixth part of BGLM, and DEHF is the sixth part of EHOP. Therefore the pyramid ABCG is equal to the pyramid DEFH: wherefore the bases, &c.

PROP. X. THEOR.—A cone is a third part of a cylinder on the same base, and of equal altitude.

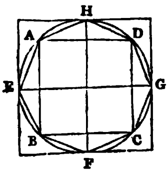
Let a cone have the same base with a cylinder, viz. the circle ABCD, and the same altitude. The cone is the third part of the cylinder; that is, the cylinder is triple of the cone.



If the cylinder be not triple of the cone, it must either be greater than the triple or less than it. First, let it be greater than the triple; and describe the square ABCD in the circle; this square is greater than the half of the circle AC, as was shown in the second proposition of this book. Upon the square ABCD form a prism of the same altitude with the cylinder; this prism is greater than half of the cylinder; because if a square be described about the circle, and a prism formed on the square, of the same altitude with the cylinder, the inscribed square is half of the circumscribed; and upon these square bases are erected parallelepipeds, viz. the prisms of the same altitude; therefore (XI. 32) the prism upon the square ABCD is the half of the prism upon the square described about the circle. But (I. ax. 9) the cylinder is less than the prism upon the square described about the circle AC: therefore the prism upon the square ABCD of the same altitude with the cylinder, is greater than half of the cylinder. Bisect the arcs AB, BC, CD, DA, in the points E, F, G, H; and join AE, EB, &c. Then each of the triangles AEB, BFC, CGD, DHA is greater than the half of the segment of the circle in which it stands, as was shown in the second proposition of this book. Describe prisms upon each of these triangles of the same altitude with the cylinder; each of these prisms is greater than half of the segment of the cylinder in which it is; because if, through the points E, F, G, H, parallels be drawn to AB, BC, CD, DA, and parallelograms be completed upon the same AB, BC, CD, DA, and parallelepipeds be formed upon the parallelograms: the prisms upon the triangles AEB, BFC, CGD, DHA are (XII. 7, cor. 2) the halves of the parallelepipeds: and (I. ax. 9) the segments of the cylinder which are upon the segments of the circle cut off by AB, BC, CD, DA are less than the parallelepipeds which contain them. Therefore the prisms upon the triangles AEB, BFC, CGD, DHA, are greater than half of the segments of the cylinder in which they are. If, therefore, each of the arcs be bisected, and straight lines be drawn from the points of division to the extremities of the arcs, and upon the triangles thus made, prisms be formed of the same altitude with the cylinder, and so on, there must at length remain some segments of the cylinder, which together (XII. lem. 1) are less than the excess of the cylinder above the triple of the cone. Let them be those upon the segments of the circle AE, EB, BF, &c. Therefore the rest of the cylinder, that is, the prism of which the base is the polygon AEBFCGDH, and of which the altitude is the same with that of the cylinder, is greater than the triple of the cone: but (XII. 7, cor. 1) this prism is triple of the pyramid upon the same base, of which the vertex is the same with the vertex of the cone: therefore the pyramid upon the base AEBFCGDH, having the same vertex with the cone, is greater than the cone, of which the base is the circle ABCD: which (I. ax. 9) is impossible.



Nor can the cylinder be less than the triple of the cone. Let it be less, if possible, and consequently the cone greater than the third part of the cylinder. In the circle AC describe a square; this square is greater than the half of the circle: and upon the square ABCD form a pyramid having the same vertex with the cone; this pyramid is greater than the half of the cone; because, as was before demonstrated, if a square be described about the circle, the square ABCD is the half of it; and if, upon these squares there be erected parallelepipeds of the same altitude with the cone, which are also prisms, the prism upon the square ABCD will be the half of that which is upon the square described about the circle; for (XI. 32) they are to one another as their bases; as are also the third parts of them. Therefore the pyramid, the base of which is the square ABCD, is half of the pyramid upon the square described about the circle. But this last pyramid is greater than the cone which it contains; therefore the pyramid upon the square ABCD, having the same vertex with the cone, is greater than the half of the cone. Bisect the arcs AB, BC, CD, DA, in the points E, F, G, H, and join AE, EB, &c. Then each of the triangles, AEB, BFC, CGD, DHA, is greater than half of the segment of the circle in which it is. Upon each of these triangles form pyramids having the same vertex with the cone. Therefore each of these pyramids is greater than the half of the segment of the cone in which it is, as was before demonstrated of the prisms and segments of the cylinder; and thus bisecting each of the arcs and joining the points of division and their extremities by straight lines, and upon the triangles erecting pyramids having their vertices the same with that of the cone, and so on, there must at length remain some segments of the cone, which together shall be less than the excess of the cone above the third part of the cylinder. Let these be the segments upon AE,

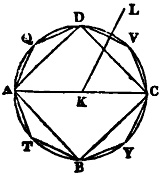


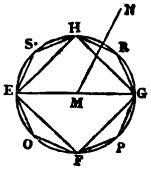
EB, BF, FC, CG, GD, DH, HA. Therefore the rest of the cone, that is, the pyramid, of which the base is the polygon AEBFCGDH, and of which the vertex is the same with that of the cone, is greater than the third part of the cylinder. But this pyramid is the third part of the prism upon the same base AEBFCGDH, and of the same altitude with the cylinder. Therefore this prism is greater than the cylinder of which the base is the circle AC; which (I. ax. 9) is impossible. Therefore the cylinder is not less than the triple of the cone: neither, as has been demonstrated, is it greater than the triple. Therefore the cylinder is triple of the cone, or the cone is the third part of the cylinder. A cone, therefore, &c.

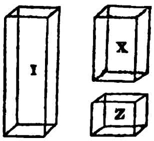
Prop. XI. Theor.—Cones and cylinders of the same altitude, are to one another as their bases.

Let the cones and cylinders, of which the bases are the circles ABCD, EFGH, and the axes KL, MN, and AC, EG, the diameters of their bases, be of the same altitude. As the circle BD to the circle FH, so is the cone AL to the cone EN.

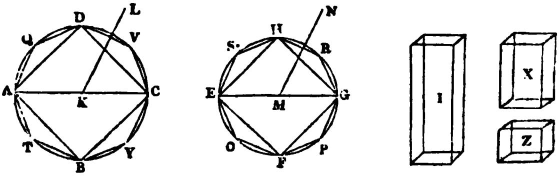
If it be not so, let BD be to FH, as the cone AL to some solid either less than the cone EN, or greater than it. First, let it be to X, a solid less than EN; and let Z be the solid which is equal to the excess of EN above X; therefore the cone EN is equal to X, Z together. In the circle FH describe the square EFGH: this square is greater than the half of the circle. Upon the square FH form a pyramid of the same altitude with the cone: this pyramid is greater than half of the cone. For, if a square be described about the circle, and a pyramid be formed upon it, having the same vertex with the cone, the pyramid inscribed in the cone is (XII. 6) half of the pyramid circumscribed about it. But the cone is less than the circumscribed pyramid; therefore the pyramid of which the base is the square EFGH, and its vertex the same with that of the cone, is greater than half of the cone. Bisect the arcs EF, FG, GH, HE, in the points O, P, R, S, and join EO, OF, &c. Then each of the triangles EOF, FPG, GRH, HSE, is greater than half of the segment of the circle in which it is. Upon each of these triangles form a pyramid having the same vertex with the cone; each of these pyramids is greater than the half of the segment of the cone in which it is: and thus dividing each of these arcs into two equal parts, and from the points of division drawing straight lines to the extremities of the arcs, and







upon each of the triangles thus made, describing pyramids having the same vertex with the cone, and so on, there must at length remain (XII. lem. 1) some segments of the cone which are together less than Z. Let these be the segments upon EO, OF, FP, PG, GR, RH, HS, SE: therefore the remainder of the cone, viz. the pyramid of which the base is the polygon EOFPGRHS, and its vertex the same with that of the cone, is greater than X.

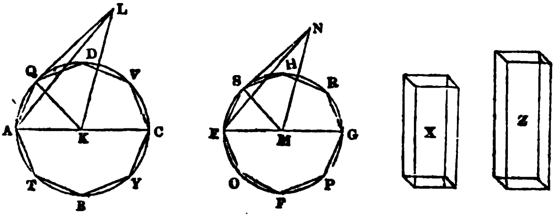


In the circle BD describe the polygon ATBYCVDQ similar to EOFPGRHS, and upon it form a pyramid having the same vertex with the cone AL. Then, because (XII. 1) the square of AC is to the square of EG, as the polygon ATBYCVDQ to the polygon EOFPGRHS; and (XII. 2) as the square of AC to the square of EG, so is the circle BD to the circle FH; therefore (V. 11) the circle BD is to the circle FH; as the polygon ATBYCVDQ to the polygon EOFPGRHS. But as the circle BD to the circle FH, so is the cone AL to X; and (XII. 6) the polygon ATBYCVDQ to the polygon EOFPGRHS, as the pyramid of which the base is the first of these polygons, and vertex L, to the pyramid of which the base is the other polygon, and its vertex N: therefore, as the cone AL to X, so is the pyramid of which the base is the polygon ATBYCVDQ, and vertex L, to the pyramid the base of which is the polygon EOFPGRHS, and vertex N. But the cone AL is greater than the pyramid contained in it; therefore (V. 14) the solid X is greater than the pyramid in the cone EN; but it is less, as was shown; which is absurd. Therefore BD is not to FH, as the cone AL to any solid which is less than the cone EN. In the same manner, it may be demonstrated that FH is not to BD, as the cone EN to any solid less than the cone AL. Nor can BD be to FH, as AL to any solid greater than EN. For, if it be possible, let it be so to the solid I, which is greater than EN. Therefore, by inversion, as FH to BD, so is I to AL. But as I to AL, so is EN to some solid, which (V. 14) must be less than AL, because I is greater than EN. Therefore, as FH is to BD, so is EN to a solid less than AL, which was shown to be impossible. BD, therefore, is not to FH, as AL is to any solid greater than EN: and it has been demonstrated that neither is BD to FH, as AL to any solid less than EN. Therefore BD is to FH, as the cone AL to the cone EN. But (V. 15) as the cone is to the cone, so is the cylinder to the cylinder, because the cylinders are triple (XII. 10) of the cones, each of each. Therefore, as the circle BD to the circle FH, so are the cylinders upon them of the same altitude. Cones and cylinders, therefore, &c.

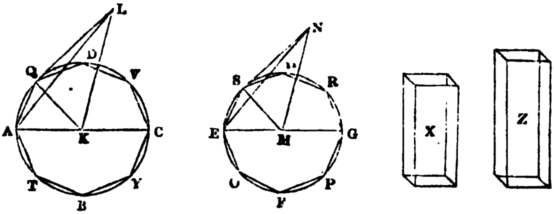
Prop. XII. Theor.—Similar cones and cylinders have to one another the triplicate ratio of that which the diameters of their bases have.

If there be similar cones and cylinders of which the bases are the circles ABCD, EFGH; the diameters of the bases AC, EG; and the axes of the cones and cylinders KL, MN: the first cone has to the second, and the first cylinder to the second, the triplicate ratio of that which AC has to EG.

For if the cone ABCDL have not to the cone EFGHN the triplicate ratio of that which AC has to EG, ABCDL will have the triplicate of that ratio to some solid which is less or greater than EFGHN. First, let it have it to a less, viz. to the solid X.



Make the same construction as in the preceding proposition, and it may be demonstrated in the same manner as in that proposition, that the pyramid, of which the base is the polygon EOFPGRHS, and vertex N, is greater than X. Describe also in the circle BD the polygon ATBYCVDQ similar to EOFPGRHS, upon which form a pyramid having the same vertex with the cone; and let LAQ, NES be two of the triangles containing those pyramids; and join KQ, MS. Then, in the triangles AKL, QKL, KA, KQ are equal, KL common, and the angles AKL, QKL equal, being (XI. def. 21) right angles: therefore (I. 4) LA, LQ are equal. For the same reason, NE, NS are equal; and therefore AL : LQ :: EN : NS. Also (XI. def. 27) because the cones are similar, AC : EG :: KL : MN; whence (V. 15) AK : EM :: KL : MN, and, alternately, AK : KL :: EM : MN; therefore (VI. 6) since the angles AKL, EMN are equal, the triangles AKL, EMN are equiangular; and (VI. 4) LA : AK :: NE : EM. Again, the angles AKQ, EMS, being each the same part of four right angles at the centres K, M, are equal; and AK : KQ :: EM : MS; therefore (VI. 6) the triangles AKQ, EMS are equiangular; and (VI. 4) AK : AQ :: EM : ES. But, as has been proved, LA : AK :: NE : EM; therefore, *ex æquo*, LA : AQ :: NE : ES; or since QL is equal to AL, and NS to NE, QL : QA :: SN : SE. Hence, (VI. 5) the triangles LQA, NSE, having the sides about all their angles proportionals, are equianglar and similar: and therefore the pyramids AKQL, EMSN are similar, because (XI. B) their solid angles are equal to one another, and they are contained by the same number of similar planes: and (XII. 8) AKQL has to EMSN the triplicate ratio of that which AK has to EM. In the same manner, if straight lines be drawn from D, V, C, &c. to K and from H, R, G, &c, to M, and pyramids be described on the triangles having the same vertices with the cones, it may be

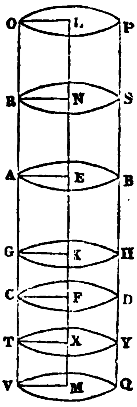


demonstrated that each pyramid in the first cone has to each in the other, taking them in the same order, the triplicate ratio of that which AK has to EM; that is, which AC has to EG. But (V. 12) as one antecedent is to its consequent, so are all the antecedents to all the consequents; therefore as the pyramid AKQL to the pyramid EMSN, so is the whole pyramid DQATBYCVL, to the whole HSEOFPGRN: and therefore also the first of these two last named pyramids has to the other the triplicate ratio of that which AC has to FG. But, by the hppothesis, the cone ABCDL has to X, the triplicate ratio of that which AC has to EG; therefore, as the cone ABCDL is to X, so is the pyramid DQATBYCVL, to the pyramid HSEOFPGRN. But the same cone is greater than the pyramid contained in it; therefore (V. 14) X is greater than the pyramid HSEOFPGRN; but it is also less; which is impossible. Therefore the cone ABCDL has not to any solid which is less than EFGHN, the triplicate ratio of that which AC has to EG. In the same manner, it may be demonstrated that neither has the cone EFGHN to any solid which is less than ABCDL, the triplicate ratio of that which EG has to AC. Nor can ABCDL have to any solid which is greater than EFGHN, the triplicate ratio of that which AC has to EG. For, if it be possible, let it have it to a greater, viz. to the solid Z. Therefore, inversely, Z has to ABCDL, the triplicate ratio of that which EG has to AC. But as Z is to ABCDL, so is EFGHN to some solid, which (V. 14) must be less than ABCDL, because Z is greater than EFGHN. Therefore EFGHN has to a solid which is less than ABCDL, the triplicate ratio of that which EG has to AC, which was demonstrated to be impossible. ABCDL, therefore, has not to any solid greater than EFGHN, the triplicate ratio of that which AC has to EG; and it was demonstrated that it could not have that ratio to any solid less than EFGHN. Therefore ABCDL has to

EFGHN, the triplicate ratio of that which AC has to EG. But (V. 15) as the cone is to the cone, so is the cylinder to the cylinder; for (XII. 10) every cone is the third part of the cylinder upon the same base, and of the same altitude. Therefore also the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG: wherefore, similar cones, &c.

PROP. XIII. THEOR.—If a cylinder be cut by a plane parallel to its bases; the plane divides the cylinder into two cylinders, which are proportional to their axes.

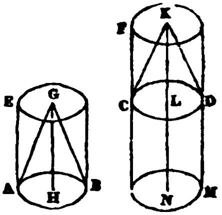
Let the cylinder AD be cut by the plane GH parallel to the bases AB, CD, meeting the axis EF in K, and let the line GH be the common section of the plane GH and the surface of the cylinder. Let AEFC be the rectangle in any position of it, by the revolution of which about EF the cylinder AD is described; and let GK be the common section of the plane GH, and the plane AF. Then, because the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common sections with it, are (XI. 16) parallel; wherefore AK is a parallelogram, and GK equal to EA, the radius of the circle AB. For the same reason, each of the straight lines drawn from the point K to the line GH may be proved to be equal to the radius of the circle AB: they are therefore all equal to one another: and (I. def. 30) the line GH is the circumference of a circle, of which the centre is K. Therefore the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the same that would be described by the revolution of the rectangles AK, GF, about EK, KF: and it is now to be shown that the cylinder AH is to the cylinder HC, as the axis EK to the axis KF.



Produce EF both ways; and take any number of straight lines EN, NL, each equal to EK, and any number FX, XM, each equal to FK; and let planes parallel to AB, CD, pass through L, N, X, M. The common sections of these planes with the cylinder continued, are circles the centres of which are the points L, N, X, M, as was proved of the plane GH; and these planes cut off the cylinders PR, RB, DT, TQ. Again, because the axes LN, NE, EK are all equal: therefore the cylinders PR, RB, BG, being to one another (XII. 11) as their bases, are equal. But because the axes LN, NE, EK are equal to one another, as also the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefore, whatever multiple the axis KL is of the axis KE, the same multiple is the cylinder PG of the cylinder GB. For the same reason, whatever multiple the axis MK is of the axis KF, the same multiple is the cylinder QG of the cylinder GD: and if KL be equal to KM, PG is equal to GQ; if KL be greater than KM, PG is greater than QG: and if less, less. Since therefore, of EK and BG there have been taken any like multiples whatever, viz. KL and PG; and of KF and GD, any like multiples whatever, viz. KM and GQ; and since it has been demonstrated, that if KL be greater than KM, PG is greater than GQ; if equal, equal; and if less, less: therefore (V. def. 5) the axis EK is to the axis KF, as the cylinder BG to the cylinder GD: wherefore, if a cylinder, &c.

PROP. XIV. THEOR.—Cones and cylinders upon equal bases are to one another as their altitudes.

Let the cylinders EB, FD be upon the equal bases AB, CD, EB is to FD, as the axis GH to the axis KL.



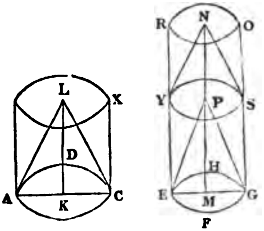
Produce KL making LN equal to GH; and let CM be a cylinder of which the base is CD, and axis LN. Then, because the cylinders EB, CM have the same altitude, they are to one another (XII. 11) as their bases: but the bases are equal; therefore also EB, CM are equal. Again, (XII. 13) because the cylinder FM is cut by the plane CD parallel to its bases, CM is to FD, as LN to KL. But CM is equal to EB, and the axis LN to GH: therefore EB is to FD, as GH to KL: and (V. 15) as EB to FD, so is the cone ABG to the cone CDK, because (XII. 10) the cylinders are triple of the cones. Therefore also GH is to KL, as the cone ABG to the cone CDK, and the cylinder EB to the cylinder FD. Wherefore, cones, &c.

PROP. XV. THEOR.—The bases and altitudes of equal cones and cylinders are reciprocally proportional; and (2) if the bases and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

1. Let the circles BD, FH, the diameters of which are AC, EG, be the bases, and KL, MN the axes, as also the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders; the bases and altitudes of the cylinders AX, EO are reciprocally proportional; that is, as BD to FH, so is MN to KL.

The altitudes MN, KL are either equal or not. First, let them be equal; and the cylinders AX, EO being also equal, and (XII. 11) cones and cylinders of the same altitude being to one another as their bases; therefore (V. A) the base BD is equal to the base FH; and as BD is to FH, so is MN to KL. But let the altitudes KL, MN be unequal, and MN the greater; from MN take MP equal to KL, and through P cut the cylinder EO by the plane YS parallel to the planes of the circles FH, RO: therefore the common section of the plane YS and the cylinder

EO is a circle, as is shown in the demonstration of the 13th proposition of this book: and consequently ES is a cylinder, the base of which is the circle EFGH, and altitude MP. Then, (V. 7) because the cylinders AX, EO are equal, as AX is to the cylinder ES, so is EO to the same ES. But (XII. 11) as AX to ES, so is the base BD to the base FH; for AX, ES are of the same altitude; and (XII. 13) as EO to ES, so is MN to MP, because EO is cut by the plane YS parallel to its bases. Therefore (V. 5) as BD to the base FH, so is MN to the altitude MP: but MP is equal to KL; wherefore as BD to FH, so is MN to KL; that is, the bases and altitudes of the equal cylinders AX, EO, are reciprocally proportional.



2. But let the bases and altitudes of the cylinders AX, EO, be reciprocally proportional, viz. BD to FH, as MN to KL; the cylinders are equal.

First, let BD be equal to FH; then because as BD is to FH, so is MN to KL, MN (V. A) is equal to KL, and therefore (XII. 11) AX is equal to EO.

But let the bases BD, FH be unequal, and let BD be the greater; and because, as BD is to FH, so is MN to KL; therefore (V. A) MN is greater than KL. Then, the same construction being made as before, because as BD to FH, so is MN to KL; and because KL is equal to MP; therefore (XII. 11) BD is to FH, as AX to ES; and as MN to MP or KL, so is EO to ES: therefore AX is to ES, as EO is to the same ES: whence the cylinder AX is equal to EO; and the same reasoning holds in cones. Therefore, the bases and altitudes, &c.

PROP. XVI. PROB.—In the greater of two concentric circles, to describe a polygon of an even number of equal sides, not meeting the less circle.*

Let ABCD, EFGH be two given circles having the same centre K; it is required to inscribe in the greater circle a polygon of an even number of equal sides, not meeting the less circle.

Through K draw the straight line BD, and from G, the point in which it meets the circumference of the less circle, draw AGC perpendicular to BD; then (III. 16) AC touches the circle FH.

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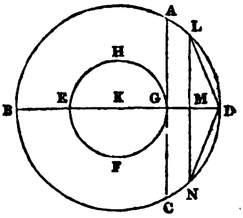
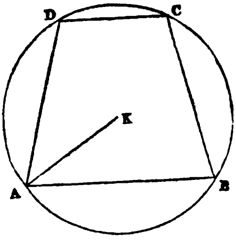
* This proposition is merely a lemma to the 17th; and it may be omitted by the student, as, in this edition, the 17th is demonstrated in such a manner as not to require it.

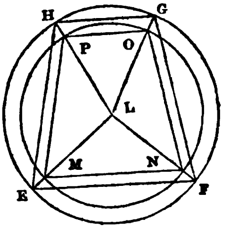


Now, if the arc BAD be bisected, and the half of it be again bisected, and so on, there must at length remain (XII. lem. 1) an arc less than AD: let this be LD; and from L draw LMN perpendicular to BD; and join LD, DN. Therefore (I. 4) LD is equal to DN; and because LN is parallel to AC, and that AC touches the circle FH; therefore LN does not meet the circle FH; and much less will the straight lines LD, DN meet it. If therefore straight lines equal to LD be applied in the circle ABCD from the point L around to N, there will be described in the circle a polygon of an even number of equal sides not meeting the less circle: which was to be done.

LEMMA. II—If two trapeziums ABCD, EFGH be inscribed in the circles, the centres of which are K, L; and if the sides AB, DC be parallel, as also EF, HG; and the other four sides AD, BC, EH, FG, be all equal; but the side AB greater than EF, and DC greater than HG: the radius KA of the circle in which the greater sides are, is greater than LE the radius of the other.

If it be possible, let KA be not greater than LE; then KA must be either equal to it, or less. First, let KA be equal to LE. Therefore, because, in two equal circles, AD, BC in the one are equal to EH, FG in the other, the arcs AD, BC are equal (III. 28) to the arcs EH, FG; but because the straight lines AB, DC are respectively greater than EF, GH, the arcs AB, DC are greater than EF, HG. Therefore the whole circumference ABCD is greater than the whole EFGH: but it is also





equal to it, which is impossible. Therefore KA is not equal to LE.

But let KA be less than LE, and make LM equal to KA, and from the centre L at the distance LM, describe the circle MNOP, meeting the straight lines LE, LF, LG, LH, in M, N, O, P; and join MN, NO, OP, PM, which (VI. 2) are respectively parallel to EF, FG, GH, HE and less than those lines. Then, because EH is greater than MP, AD is greater than MP; and the circles ABCD, MNOP are equal; therefore the arc AD is greater than the arc MP. For the same reason, the arc BC is greater than the arc NO; and because the straight line AB is greater than EF, which is greater than MN, much more is AB greater than MN. Therefore the arc AB is greater than the arc MN; and, for the same reason, the arc DC is greater than the arc PO. Therefore the whole circumference ABCD is greater than the whole MNOP; but it is likewise equal to it, which is impossible. Therefore KA is not less than LE; nor is it equal to it; it must therefore be greater. If two trapeziums, therefore, &c.

Cor. Also, if there be an isosceles triangle, the sides of which are equal to AD, BC, but its base less than AB, the greater of the parallel sides; the straight line KA may, in the same manner, be demonstrated to be greater than the straight line drawn from the centre to the circumference of the circle described about the triangle.

PROP. XVII. THEOR.—In the greater of two concentric spheres, to describe a polyhedron, the surface of which will not meet the less sphere.

Let there be two spheres about the same centre A; it is required to describe in the greater a polyhedron, the surface of which will not meet the less.

If the spheres be cut by a plane passing through the centre, the common sections of it and the spheres will be circles; because the sphere is described by the revolution of a semicircle about the diameter remaining fixed; so that in whatever position the semicircle is conceived to be, the common section of the plane in which it is, with the surface of the sphere, is the circumference of a circle: and this is a great circle* of the sphere, because (III. 15) the diameter of the sphere, which is likewise the diameter of the circle, is greater than any chord in the circle or sphere. Let then the circle made by the section of the plane with the greater sphere be BCDE, and with the less be FGH; draw the perpendicular diameters BD, CE: and draw GU perpendicular to AG, and join AU. Then, if the arc BE be bisected in L, its halves in M, K, and so on, an arc will at length be obtained, less than the arc which would have a chord equal to GU. Let BK be this arc: then the chord BK is less than GU. Draw the diameter KAN, and through A draw AX perpendicular to the plane of the circle BCDE, meeting the surface of the sphere in X. Let planes pass through AX and each of the straight lines BD, KN: these, from what has been said, will produce great circles on the surface of the sphere; let BXD, KXN be the semicircles thus made upon the diameters BD, KN. Then (XI. 18) because XA is at right angles to the plane of the circle BCDE, the semicircles BXD, KXN are at right angles to that plane. And because the semicircles BED, BXD, KXN, upon the equal diameters BD, KN, are equal, their halves BE, BX, KX are also equal. Therefore, as many arcs as are in BE, so many there are in BX, KX equal to the arcs BK, KL, LM, ME. Let these arcs be BO, OP, PR, RX; KS, ST, TY, YX; draw their chords, and join OS, PT, RY: draw also OV, SQ perpendiculars to AB, AK: and because the plane BOXD is at right angles to the plane BCDE, and in one of them BOXD, OV is drawn perpendicular to AB their common section, therefore (XI. def. 3) OV is perpendicular to the plane BCDE. For the same reason, SQ is perpendicular to BCDE, because the plane KSXN is perpendicular to it. Join VQ; and because in the equal semicircles BXD, KXN, the arcs BO, KS are equal, the angles BAO, KAS are equal; and (const.) AVO, AQS are right angles; therefore (I. 26) OV is equal to SQ, and AV equal to AQ. But the whole BA is equal to the whole KA, therefore the remainders BV, KQ are equal. As, therefore, BV : VA :: KQ : QA; wherefore (VI. 2) VQ is parallel to BK. Also (XI. 6) because OV, SQ are each perpendicular to the plane of the circle BCDE, they are parallel: and they have been proved to be equal; therefore QV, SO are (I. 33) equal and parallel: and because QV is parallel to SO, and also to KB; OS is parallel (XI. 9) to BK; and BO, KS, which join them, are in the same plane in which these parallels are, and the quadrilateral KBOS is in one plane. And if PB,

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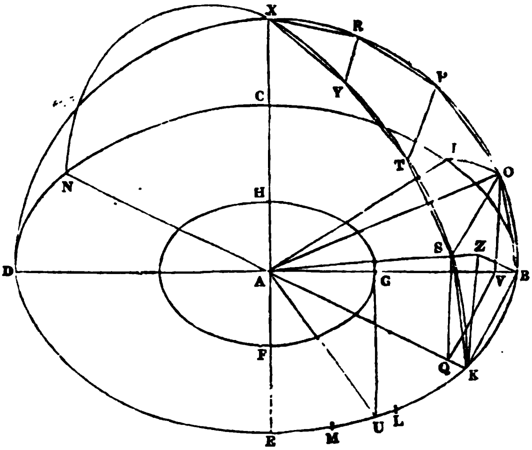
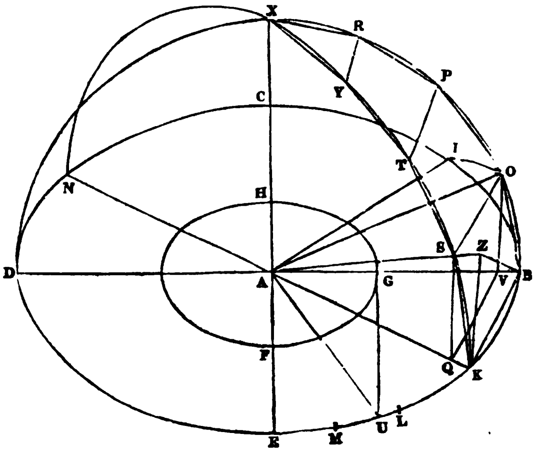
* See Appendix, IV. 10.



TK be joined, and perpendiculars be drawn from the points P, T, to the straight lines AB, AK, it may be demonstrated, that TP is parallel to KB, in the same way that SO was shown to be parallel to the same KB; wherefore (XI. 9) TP is parallel to SO, and the quadrilateral SOPT is in one plane. For the same reason, the quadrilateral TPRY is in one plane: and (XI. 2) the triangle YRX is also in one plane. If, therefore, from the points O, S, P, T, R, Y, straight lines be drawn to the centre A, there will be formed a polyhedron between the arcs BX, KX, composed of pyramids, the bases of which are the quadrilaterals KBOS, SOPT, TPRY, and the triangle YRX, and of which the common vertex is A: and if the same construction be made upon each of the chords of KL, LM, ME, as has been done upon BK, and the like be done also in the other three quadrants, and in the other hemisphere; there will be formed a polyhedron inscribed in the sphere, composed of pyramids, the bases of which are the quadrilaterals already mentioned, and the triangle YRX, and those formed in the same manner in the rest of the sphere, the common vertex of them all being A.

Also, the surface of this polyhedron does not meet the less sphere. For, from A draw (XI. 11) AZ perpendicular to the plane of the quadrilateral KBOS, meeting it in Z, and join BZ, ZK. Then, because AZ is perpendicular to the plane KBOS, it is perpendicular to BZ and ZK: and (I. 47, cor. 5) because AB is equal to AK, and AZ common to the two right-angled triangles AZB, AZK; ZB is equal to ZK. In the same manner it may be demonstrated, that the straight lines drawn from Z to O, S, are equal to BZ or ZK. Therefore the circle described from the centre Z, at the distance ZB, will pass through the points K, O, S, and KBOS will be a quadrilateral in the circle. Now, because KB is greater than QV, and QV equal to SO, therefore KB is greater than SO: but KB is equal to each of the straight lines BO, KS; wherefore each of the arcs cut off by KB, BO, KS, is greater than that cut off by OS: and these three arcs, together with a fourth equal to one of them, are greater than the same three together with that cut off by OS; that is, than the whole circumference of the circle; therefore the arc subtended by KB is greater than the fourth part of the whole circumference of the circle KBOS, and consequently the angle BZK at the centre is obtuse: and (I. 19) BK is greater than BZ. But it has been seen already that GU is greater than BK: much more then is GU greater than BZ. Therefore, in the two right-angled triangles AGU, AZB, the hypotenuses AU, AB are (I. def. 30) equal; but the leg GU is greater than the leg BZ: hence (I. 47, cor. 5) the remaining leg AZ is greater than the remaining leg AG. But AZ is perpendicular to the plane KBOS, and is therefore the shortest of all the straight lines that can be drawn from A, the centre of the sphere, to that plane. Therefore the plane KBOS does not meet the less sphere.

And that the other planes between the quadrants BX, KX, fall without the less sphere, is thus demonstrated. From the point A draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and, as was demonstrated of the plane KBOS and the point Z, in the same way it may be shown, that the point I is the centre of a circle described about SOPT, and that OS is greater than PT: and PT was shown to be parallel to OS. Therefore, because the two trapeziums KBOS, SOPT inscribed in circles, have their sides KB, OS parallel, as also OS, PT; and their other sides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS than PT; therefore (XII. lem. 2) ZB is greater than IO. Join AO, which will be equal to AB. Then, in the right-angled triangles AIO, AZB, the hypotenuses AO, AB are equal, but the leg ZB is greater than the leg IO: wherefore (I. 47, cor. 5) the leg AI is greater than the leg AZ. But it has been proved that AZ is greater than AG; much more then is AI greater than AG. Therefore the plane SOPT falls wholly without the less sphere. In the same manner it may be demonstrated, that the plane TPRY falls without the same sphere, as also the triangle YRX, viz. by the corollary to the second lemma. And in a similar way it may be demonstrated, that all the planes which contain the polyhedron, fall without the interior sphere. Therefore, in the greater of the two concentric spheres, a polyhedron is described, the surface of which does not meet the less: which was to be done.

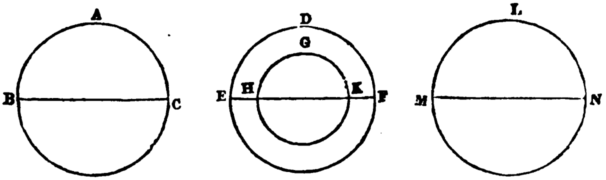


Cor. If in the less sphere a polyhedron be described by drawing straight lines between the points in which the radii of the sphere, drawn to all the angles of the polyhedron in the greater sphere, meet the surface of the less; in the same order in which the points are joined in which the same radii meet the surface of the greater sphere; the polyhedron in the sphere BCDE has to this other polyhedron the triplicate ratio of that which the diameter of the greater sphere has to the diameter of the less. For, if these two solids be divided into the same number of pyramids, and in the same order; the pyramids will be similar, each to each: because they have the solid angles at their common vertex, the centre of the sphere, the same in each pyramid, and (XI. B) their other solid angles at the bases equal to one another, each to each, because they are contained by three plane angles, equal each to each; and the pyramids are contained by the same number of similar planes; and (XI. def. 8) are therefore similar to one another, each to each. But (XII. 8, cor.) similar pyramids have to one another the triplicate ratio of their homologous sides. Therefore the pyramid, of which the base is the quadrilateral KBOS, and the vertex A, has to the pyramid in the other sphere of the same order the triplicate ratio of their homologous sides; that is, of that ratio which the radius of the greater sphere has to the radius of the less. In like manner, each pyramid in the greater sphere has to each of the same order in the less, the triplicate ratio now mentioned. And as one antecedent is to its consequent, so are all the antecedents to all the consequents: wherefore the whole polyhedron in the greater sphere has to the whole polyhedron in the less, the triplicate ratio of that which AB, the radius of the first, has to the radius of the other; that is which the diameter BD of the greater has to the diameter HF of the less.

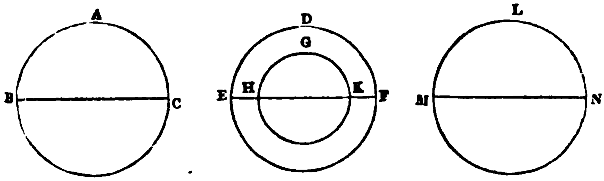
Prop. XVIII. Theor.—Spheres have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two spheres, of which the diameters are BC, EF: the sphere ABC has to the sphere DEF the triplicate ratio of that which BC has to EF.

For, if not, the sphere ABC will have to a sphere either less or greater than DEF, the triplicate ratio of that which BC has to EF. First, let it have that ratio to a less, viz. to the sphere



GHK; and let DEF have the same centre with GHK; and (XII. 17) in the greater DEF describe a polyhedron, the surface of which does not meet the less, and in ABC describe another similar to that in DEF. Therefore (XII. 17, cor.) the polyhedron in ABC has to the polyhedron in DEF, the triplicate ratio of that which BC has to EF. But ABC has to GHK, the triplicate ratio of that which BC has to EF: therefore (V. 11) as ABC to GHK, so is the polyhedron in ABC to the polyhedron in DEF. But the sphere ABC is greater than the polyhedron in it; therefore, also, (V. 14) the sphere GHK is greater than the polyhedron in the sphere DEF; but it is also less, because it is contained within it; which is impossible. Therefore ABC has not to any sphere less than DEF, the triplicate ratio of that which BC has to EF. In the same manner it may be demonstrated, that DEF has not to any sphere less than ABC, the triplicate ratio of that which EF has to BC. Nor can ABC



have to any sphere greater than DEF, the triplicate ratio of that which BC has to EF. For, if it can, let it have that ratio to a greater sphere LMN. Then, by inversion, LMN has to ABC, the triplicate ratio of that which EF has to BC. But as LMN to ABC, so is DEF to some sphere, which (V. 14) must be less than ABC, because LMN is greater than DEF. Therefore DEF has to a sphere less than ABC the triplicate ratio of that which EF has to BC; which was shown to be impossible. ABC, therefore, has not to any sphere greater than DEF, the triplicate ratio of that which BC has to EF: and it was demonstrated, that it has not that ratio to any sphere less than DEF. Therefore the sphere ABC has to DEF, the triplicate ratio of that which BC has to EF: wherefore spheres, &c.

APPENDIX.

APPENDIX

TO THE

FIRST SIX, AND THE ELEVENTH AND TWELFTH BOOKS

OF

EUCLID'S ELEMENTS.

BOOK I.*

DEFINITIONS.

1. *Lines*, *angles*, and *spaces* are said to be *given in magnitude*, when they are either exhibited, or when the method of finding them is known.

2. *Points*, *lines*, and *spaces* are said to be *given in position*, which have always the same situation, and which are actually exhibited, or can be found.

3. A *circle* is said to be *given in magnitude*, when its radius is given; and *in position*, when its centre is given.

Magnitudes, instead of being said to be *given in magnitude*, or *given in position*, are often said simply to be *given*, when no ambiguity arises from the omission.

4. A *ratio* is said to be *given*, when it is the same as that of two given magnitudes.

5. A *rectilineal figure* is said to be *given in species*, when its several angles, and the ratios of its sides are given.

6. When a series of unequal magnitudes, unlimited in number, agree in certain relations, the greatest of them is called a *maximum*; the least a *minimum*.

Thus, of chords in a given circle, the diameter is the maximum; and of straight lines drawn to a given straight line, from a given point without it, the minimum line is the perpendicular.

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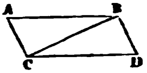
* This book contains a number of miscellaneous propositions, several of which are curious and important. The first five demonstrate the converses of the 34th proposition in the first book of Euclid, and the 11th, 12th, 21st, 22d, and 35th propositions, and the corollary to the 36th, in the third book. They are often used in demonstrating other propositions.

PROPOSITIONS.

Prop. I. Theor.—**If the opposite sides, or the opposite angles of a quadrilateral be equal, it is a parallelogram.**

1. Let AD be a quadrilateral; and first, let AB be equal to CD, and AC to BD: AD is a parallelogram.

Join BC. Then in the triangles ABC, DCB, AB, AC are equal to DC, DB, each to each, and BC is common: therefore (I. 8) the angle ABC is equal to DCB, and ACB to DBC; and (I. 27) AB is parallel to CD, and AC to BD. Hence (I. def. 24) AD is a parallelogram.



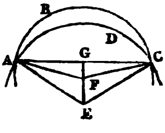
2. Again, if the angle BAC be equal to BDC, and ABD to ACD, AD is a parallelogram.

For (I. ax. 2) the angles CAB, ABD are equal to BDC, DCA; and therefore, all four being equal (I. 32, cor. 1) to four right angles, CAB, ABD are equal to two right angles, and (I. 28) AC, BD are parallel. In the same manner it would be proved that AB, CD are parallel: and therefore (I. def. 24) AD is a parallelogram.

Prop. II. Theor.—**If two circles cut one another, the straight line passing through their centres bisects their common chord.**

Let the circles ABC, ADC cut one another in A and C; join AC, and through the centres E, F, draw EFG; AC is bisected in G.

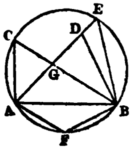
Join EA, EC, FA, FC. Then the triangles AEF, CEF have the sides EA, AF equal to EC, CF, each to each, and EF common: therefore (I. 8) the angles AEF, CEF are equal. Again, the triangles AEG, CEG have EA, EC equal, EG common, and the contained angles equal: therefore AG, GC are equal.



Cor. Hence, when the circles cut one another, the straight line passing through their centres does not pass through either point of intersection: and therefore, if the line passing through the centres pass also through a point in which the circumferences meet, the circles must touch one another in that point.

Prop. III. Theor.—**If two triangles on the same base, and on the same side of it, have equal vertical angles, the vertex of each is in the circumference of the circle described about the other.**

Let ABC be a triangle inscribed in the circle CAB: any other triangle on the base AB, and having its vertical angle equal to C, will have its vertex on the arc ACB.



If possible, let ADB be a triangle on the base AB, and having the angle ADB equal to C, but the vertex D not on the arc ACB. Let AD, produced if necessary, meet the circumference in E, and join BE. Then (III. 21) the angle AEB is equal to C: wherefore ADB is equal to AEB, which (I. 16) is impossible. Therefore, if the angle ADB be equal to C, D must fall on the arc ACB.

PROP. IV. THEOR.—**If two opposite angles of a quadrilateral be together equal to two right angles, a circle may be described about it.**

Let ADBF (see the figure for the preceding proposition) be a quadrilateral having the angles ADB, AFB together equal to two right angles: a circle described through A, F, B will also pass through D.

For, if possible, let D be within the circle passing through A, F, B; produce AD to E, and join BE. Then (III. 22) the angles E, F are equal to two right angles, and therefore to ADB and F. Take away F, and there remains ADB equal to E, which (I. 16) is impossible. In the same manner, if D were supposed to be without the circle, it would be shown that the angle at D would be equal to AEB, which (I. 16) is also impossible. Therefore D must be on the circumference.

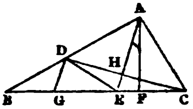
PROP. V. THEOR.—**If two straight lines intersect each other, so that the rectangle under the segments of one of them is equal to the rectangle under the segments of the other, their extremities lie in the circumference of a circle.**

Let the straight lines BC, AD (see the figure for prop. 3) cut one another in G, and let the rectangles AG.GD, BG.GC be equal; a circle described through A, B, C will also pass through D.

For, if possible, let D not fall on the circumference; and let AD, produced, if necessary, meet the circumference in E. Then (III. 35) the rectangle AG.GE is equal to BG.GC, and therefore to AG.GD; which (VI. 1) is impossible, unless D and E coincide: and therefore D must be on the circumference.

Schol. By means of the corollary to the 36th proposition of the third book, it would be shown in a similar manner, that if two straight lines meet in a point, and if they be so divided, that the rectangle under one of them and its segment next the common point, is equal to the rectangle under the other, and its corresponding segment, the points of section and the extremities remote from the common point lie in the circumference of the same circle.

PROP. VI. THEOR.—From AB, the greater side of the triangle ABC, cut off AD equal to AC, and join DC: draw AE bisecting the vertical angle BAC, and join DE: draw also AF perpendicular to BC, and DG parallel to AE. Then (1.) the angle DEB is equal to the difference of the angles at the base, ACB, ABC; or of BAF, CAF, or of AEB, AEC: and DE is equal to EC: (2) the angles BCD, EAF are each equal to half the same difference: (3) ADC or ACD is equal to half the sum of the angles at the base, or to the complement of half the vertical angle: (4) BG is equal to the difference of the segments BE, EC, made by the line bisecting the vertical angle.



1. In the triangles AED, AEC, AD, AC are equal, AE common, and the contained angles equal: therefore (I. 4) DE is equal to EC, the angle ADE to ACE, and AED to AEC. But (I. 32) becaus BD is produced, the angle ADE is equal to B and BED: therefore BED is the difference of B and ADE, or of B and ACB. Also BED is the difference of AEB, AED, or of AEB, AEC. Again, ABF, BAF are equal to ACF, CAF, each pair being (I. 32) cor. 3) equal to a right angle. Take away ABF: then, because the difference of ACF, ABF is BED, there remains BAF equal to BED, CAF; that is, BED is the difference of BAF, CAF.

2. The difference BED is equal (I. 32) to the two angles ECD, EDC, which (I. 5) are equal: therefore ECD is half of BED. Again, in the triangles AHD, AHC, because AD, AC are equal, AH common, and the contained angles equal, DH is equal (I. 4) to HC, and the angles at H are equal, and are therefore right angles. Then, in the triangles AEF, CEH, the angles AFE, CHE are equal, and AEC common; therefore (I. 32, cor. 5) the angle EAF is equal to ECH, which has been proved to be equal to the half of BED.

3. Since the angle BAC is common to the triangles ABC, ADC, the angles ADC, ACD are (I. 32) equal to ABC, ACB; and therefore, since ADC, ACD are equal, each of them is equal to half the sum of ABC, ACB; also either of them, ADC, is the complement of DAH, half the vertical angle, since AHD is a right angle.

4. Because * DH, HC are equal, and HE, DG parallel, GE is

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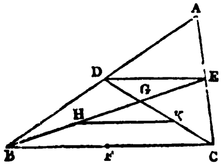
* It is easy to prove without proportion, that if AB (see the figure for the next proposition) be bisected in D, the straight line DE parallel to BC bisects AC, and that the triangle ADE is a fourth of ABC. For (I. 37) the triangles BDC, BEC are equal. But (I. 38) BDC is half of ABC; and therefore BEC is half of ABC, and is equal to BEA. Hence the bases AE, EC are equal; for if they were not, the triangles ABE, CBE (I. 38. cor. 1) would be unequal. Again, because AD, DB are equal, the triangle ADE is (I. 38) half of ABE, and therefore a fourth of ABC.

equal (VI. 2) to EC; and therefore BG, the difference of BE, GE, is also the difference of BE, EC.*

Prop. VII. Theor.—In a triangle, straight lines drawn from the points of bisection of the three sides to the opposite angles, all pass through the same point.

Let D, E, F be the points of bisection of the sides of the triangle ABC, and draw BE, CD, intersecting in G; bisect BG, CG in H, K, and join DE, HK. Then DE, HK are parallel and equal, each of them† being parallel to BC, and equal to half of it. Hence, the triangles DGE, KGH are equiangular, and (I. 26) HG is equal to GE, and KG to GD: and therefore BG is double of GE, and CG of GD. In the same manner, it would be proved, that a straight line joining AF would divide BE into two segments, of which the one terminated in B would be double of the one terminated in E; that is, the straight line AF would pass through G.



Schol. It appears from the demonstration, that each of the straight lines AF, BE, CD is divided at their common point into two segments, of which the one drawn to the angle of the triangle is double of the other.

Cor. 1. Suppose HC were joined; then the three triangles CBH, CHG, CGE are (I. 38) all equal; and therefore BGC, being two thirds of BEC, is one third of BAC, which is double of BEC. Straight lines therefore drawn from G to the three angles, trisect the triangle ABC.

Cor. 2. Hence the problem in which it is required to draw a straight line through a given point D, so that the part of it between two given straight lines AC, BC may be bisected in D, is solved simply by drawing DE parallel to BC, making EA equal to EC, and drawing ADB; or should the ratio of AD to DB not be that of equality, the solution will be the same, except that AE must have that ratio to EC.

Conversely, if DE bisect AB, AC, it is parallel to BC. For (I. 38, cor. 2) the triangles BDC, BEC are each half of ABC; and these being therefore equal, DE is parallel (I. 39) to BC.

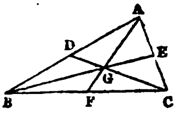
Hence, it is plain (I. 26), that the straight lines joining DE, DF, EF, divide the triangle ABC into four equal triangles, similar to the whole and to one another; and that each of these lines is equal to half the side to which it is parallel.

* Instead of cutting off AD equal to AC, AC may be produced through C, and by cutting off, on AC thus produced, a part terminated at A, and equal to AB, and by making a construction similar to that of the foregoing proposition, it will be easy to establish the same properties as those above demonstrated, or ones exactly analogous.

† By the note to the preceding proposition; or by VI. 2, and 4.

Prop. VIII. Theor.—Four times the sum of the squares of the three straight lines drawn from the angles of a triangle to the points of bisection of the opposite sides, are equal to three times the sum of the squares of the sides.

Let AF, BE, CD bisect the sides of a triangle ABC in the points F, E, D: four times the sum of the squares of AF, BE, CD, are equal to three times the sum of the squares of AB, BC, AC.

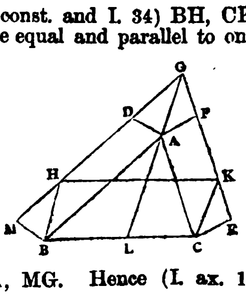


For (II. A) $AB^2+AC^2=2BF^2+2AF^2$. From this, by doubling, and since (II. 4, cor. 2) $4BF^2=BC^2$, there arises $2AB^2+2AC^2=BC^2+4AF^2$. For the same reason, $2AB^2+2BC^2=AC^2+4BE^2$, and $2AC^2+2BC^2=AB^2+4CD^2$. Add these equals together, and from the sums take BC^2, AC^2, AB^2: then $3AB^2+3BC^2+3AC^2=4AF^2+4BE^2+4CD^2$.

Cor. Since (App. I. 7, schol.) AF is treble of GF, the square of AF (I. 46, cor. 4) is equal to nine times the square of GF; and therefore the squares of AF, BE, CD are evidently equal to nine times the squares of GF, GE, GD. Hence, by this proposition, thirty-six times the squares of GF, GE, GD are equal to three times the squares of AB, BC, AC; and consequently *the squares of* GF, GE, GD, *are equal to one twelfth of the squares of the sides* AB, BC, CA. Hence also, since AG, BG, CG are double of GF, GE, GD, each of each, the squares of the former are quadruple of the squares of the latter; and hence *the sum of the squares of* AG, BG, CG *is one third of the sum of the squares of the three sides* AB, BC, AC.

Prop. IX. Theor.—Let ABC be any triangle, and on AB, AC, two of its sides, let any parallelograms ABMD, ACEF, be described: produce MD, EF to meet in G, and join GA; draw BH, CK parallel to AG, and meeting MG, EG, in H and K; join HK: HBCK is a parallelogram, and is equal to the two ABMD, ACEF.

Produce GA to L. Then, since (const. and I. 34) BH, CK are parallel and equal to AG, they are equal and parallel to one another: therefore also (I. 33) HK is equal and parallel to BC, and (I. def. 24) BCKH is a parallelogram; and LH, LK are also parallelograms. Now (I. 35) the parallelograms HL, HBAG are equal, because they are on the same base BH, and between the same parallels BH, LG: and the parallelograms HBAG, MBAD are equal, because they are on the same base AB, and between the parallels BA, MG. Hence (I. ax. 1)

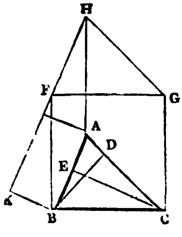


MBAD is equal to HL. In the same manner, it would be shown that ACEF is equal to LK: and therefore (I. ax. 2) the whole parallelogram BK is equal to the two MBAD, ECAF.

Schol. The 47th proposition of the first book is the case of this proposition in which BAC is a right angle, and the parallelograms AM, AE squares (for which case the student may make a diagram for himself); as in that case, HBCK is also a square. For then DF is a parallelogram; and in the triangles GFA, BAC, right-angled at F and A, FA is equal to AC, and GF to BA, each being equal to DA: therefore (I. 4) BC is equal to AG, or to CK, which is equal to AG: and the angle ACB is equal to FAG, or to ECK, which (I. 34, cor. 4) is equal to FAG. To the equal angles ACB, ECK, add KCA: then BCK is equal to ACE, which is a right angle. Hence the parallelogram HBCK is a square.

PROP. X. THEOR.—From B, C, the extremities of one of the sides of any triangle ABC, draw perpendiculars to the other sides, meeting those sides, or their continuations, in D and E; and on BC describe the square BG: BG is equal to the rectangles AB.BE, AC.CD; that is, to the rectangles contained by the sides and the segments intercepted on them, between the perpendiculars and the third side.

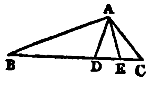
Complete the parallelograms ABFH, ACGH; produce HF, and complete the rectangle AK. Then (I. 34, cor. 4) the triangles ABC, HFG are equiangular, and (I. 26) they are equal, since the sides BC, FG are equal. Take these triangles separately from the figure BCGHF, and there remains the square BG equal to the parallelograms AF, AG: and (I. 35) the parallelograms AF, AK are equal. From the right angles KBE, FBC take FBA, and the remaining angles KBF, EBC are equal. Hence, since the angles at K and E are right angles, the triangles BFK, BCE are equiangular; and the sides BF, BC are equal: hence (I. 26) BK is equal to BE. Therefore the rectangle AK is contained by AB, BK, or by AB, BE; and it has been proved to be equal to AF. In the same manner it would be shown that the rectangle AC.CD is equal to the parallelogram AG. Therefore the rectangles AB.BE and AC.CD are together equal to the parallelograms AF, AG; that is, as has been proved, to the square BG.



Schol. If BAC be a right angle, the perpendiculars coincide with BA, CA, and the rectangles AB.BE, AC.CD become the squares of AB, AC: whence it appears that the 47th proposition of the first book, is also a case of this proposition.

PROP. XI. THEOR.—Let ABC be any triangle; and from one of the angles let straight lines AD, AE be drawn to the opposite side, making with the other sides the angles BAD, CAE respectively equal to C, B, the remote angles at the base: the square of AB is equal to the rectangle CB.BD; and the square of AC to the rectangle BC.CE.

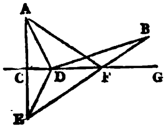
For the triangles ABC, ABD have the angle B common and the angle C equal to BAD; therefore (VI. 4, cor.) BC : BA :: BA : BD. Hence, (VI. 17) the square of BA is equal to the rectangle BC.BD: and it would be proved in a similar manner, that the square of AC is equal to the rectangle BC.CE.



Schol. If BAC be a right angle, the angles B, C being together equal to it, AD, AE will coincide, and the segments BD, EC will make up BC: and (II. 2) the squares of AB, AC will be equal to the square of BC. This, therefore, is another proposition of which the 47th of the first book is a particular case.

PROP. XII. THEOR.—If through two given points two straight lines be drawn to meet in a given straight line, and to make equal angles with it, their sum is less than the sum of any other straight lines drawn through the given points, and meeting in the given line.

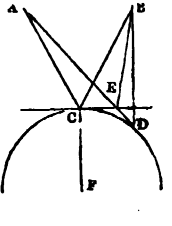
Let A, B be the given points, and CG the given line: draw ACE perpendicular to CG, making CE equal to CA, and draw EB cutting CG in F; join also AF: then (I. 4) FA, FE are equal, and the angle AFC is equal to EFC, or (I. 15) to BFG, so that AF, BF make equal angles with CG: AF, FB are together less than any other straight lines AD, DB drawn from A and B to meet in CG. Join DE. Then (I. 4) DE is equal to DA: add DB to each, and ED, DB are equal to AD, DB. But (I. 20) ED, DB are greater than EFB, that is, than AF, FB, because EF, AF are equal: therefore also AD, DB are greater than AF, FB: and in the same manner it may be shown that AF, FB are less than two straight lines drawn to any other point in CG.



If the given points be B and E, on the opposite sides of CG, it is plain that EF, FB are the shortest lines, and that they make equal angles with CG.

PROP. XIII. THEOR.—The sum of two straight lines drawn from two given points, without a given circle, and meeting in a convex circumference, is a minimum, when they make equal angles with the tangent at that point.

Let A, B be two points without a given circle, and C a point in the convex circumference, such that the straight lines CA, CB make equal angles with CE, the tangent at C: CA, CB are together less than the sum of two straight lines DA, DB drawn from any other point D in the circumference to A, B.



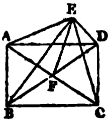
Let DA cut the tangent in E, and join EB. Then ED, DB are greater than EB. Add AE; then AD, DB are greater than AE, EB. But (App. I. 12) AE, EB are greater than AC, CB, because AC, CB make equal angles with CE: much more, therefore, are AC, CB less than AD, DB.

Cor. Since the tangent makes equal angles with the radius, the angles FCA, FCB are equal, when the sum of AC, CB is a minimum.

Prop. XIV. Theor.—The squares of the straight lines drawn from any point to two opposite angles of a rectangle, are together equal to the squares of those drawn to the other angles.

Let ABCD be a rectangle, and E a point, either within the figure or without it; join EA, EC; EB, ED: the squares of EA, EC are together equal to the squares of EB, ED.

Draw the diagonals intersecting in F, and join EF. Then, in the triangles ABC, DCB, AB, DC are equal, BC common, and the right angles at B and C equal; therefore (I. 4) the diagonals AC, BD are equal: and since (II. B, cor.) the diagonals bisect each other, the lines AF, BF, CF, DF are all equal. Now (II. A) the squares of AE, EC are double of the squares of AF, FE. For the same reason, the squares of BE, ED are double of the squares of BF, FE, or of AF, FE. Therefore (I. ax. 1) the squares of AE, EC are equal to the squares of BE, ED.

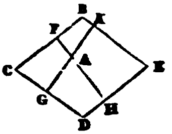


Schol. This proposition holds, whether the point E be in the plane of the rectangle or not; and the demonstration is the same in both cases.

Prop. XV. Theor.—The perpendiculars drawn from any given point within an equilateral figure to the several sides, or the sides produced, are together equal to the sum of those drawn from any other point within the figure to the sides, or their continuations.

Let A be any point within the equilateral figure BCDE, and let AF, AG, AH, AK be drawn perpendicular to the sides: the sum of these perpendiculars is equal to the sum of the perpendiculars drawn from any other point within the figure to the sides.

For, if straight lines were drawn from A to the several angles, the rectangle BC.AF would be double of the triangle ABC; the restangle CD.AG, or BC.AG, double of CAD, &c. Hence (II. 1) the rectangle under BC and the sum of the perpendiculars AF, AG, AH, &c. is double of the whole figure BCDE. In the same manner it would be shown, that the rectangle under BC and the sum of the perpendiculars drawn from any other point within the figure to the sides, is double of the figure BCDE. Hence the rectangle under BC and the sum of the one system of perpendiculars, is equal to the rectangle under BC and the sum of the others. The sums therefore are equal; for if they were unequal, the rectangle under BC and one of them could not be equal to the rectangle under BC and the other.*



Cor. 1. Since the rectangle under the side and perpendicular of an equilateral triangle is double of the triangle, it follows that the perpendicular is equal to the sum of the three perpendiculars drawn from any point within the triangle to the sides.

Cor. 2. In an equilateral and equiangular figure, the sum of the perpendiculars drawn from any point within it to the sides, is equal to the radius of the inscribed circle taken as often as the figure has sides.

Cor. 3. The sum of the two perpendiculars drawn from any point in the base of an isosceles triangle to the other sides, is equal to the sum of those drawn from any other point in the base to those sides, or to the perpendicular drawn from one of the equal angles to the opposite side. The method of proof is the same as that of the proposition, if the vertex and the point in the base be joined.

PROP. XVI. THEOR.—The squares of two straight lines drawn from any point in a diameter of a circle, or in its continuation, to the extremities of a chord parallel to it, are together equal to the squares of the segments between the point and the extremities of the diameter.

Let BDC be a circle, A a point in the diameter BC, or its continuation, and DE a chord parallel to BC; join AD, AE: the squares of AD, AE are together equal to the squares of AB, AC.

Through the centre F draw EFG, and join FD, AG. Then the angles AFD, AFG are equal, because (I. 29) they are equal to the equal angles FDE, FED. Therefore, in the triangles

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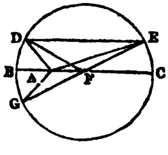
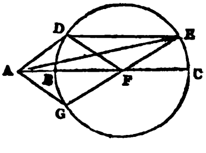
* If the given point were *without* the equilateral figure, it would be shown in a similar manner, that if the sum of the perpendiculars which fall on sides lying between the point and the figure, be taken from the sum of the others, the remainder is equal to the sum of the perpendiculars to the sides from a point within the figure. It is easy to see how this will modify the corollaries to this proposition.

AFD, AFG, since FD is equal to FG, FA common, and the contained angles equal; AD, AG (I. 4) are equal. Now (II. A) because GF, FE are equal, the squares of GA, AE, or of DA, AE are equal to twice the square of GF or BF, and twice the square of AF. But (II. 9 or 10) the squares of BA, AC are equal to twice the squares of BF, AF. Therefore (I. ax. 1) the squares of DA, AE are equal to the squares of BA, AC.





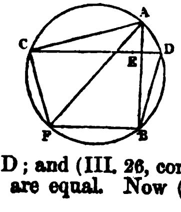
Cor. 1. Hence (I. ax. 1) if there were another chord parallel to DE, the squares of the straight lines drawn from its extremities to A are equal to the squares of AD, AE; and the same holds regarding all other chords parallel to the diameter through A.

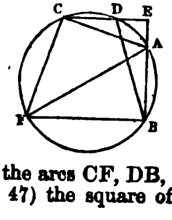
Cor. 2. The squares of BA, AC are equal to twice the square of the line drawn from A to the point of bisection of the arc DE. This may be demonstrated in the same manner as the proposition itself: and the line so drawn may be regarded as what AD, AE become, when DE recedes from the diameter so as to be reduced to a point.

PROP. XVII. THEOR.—If the circumference of a circle be cut by two straight lines which are perpendicular to one another, the squares of the four segments between the point of intersection of the two lines and the points in which they meet the circumference, are together equal to the square of the diameter.

Let the straight lines AB, CD cut one another perpendicularly in the point E, and the circumference of the circle ABC in the points A, B, C, D; and draw the diameter AF: the sum of the squares of AE, BE, CE, DE is equal to the square of AF.

Join AC, CF, FB, BD. Then (III. 31) ACF, ABF are right angles; and therefore, AEC being a right angle, FB (I. 28) is parallel to CD; and (III. 26, cor.) the arcs CF, DB, and (III. 29) their chords are equal. Now (I. 47) the square of BD or CF





is equal to the squares of BE, ED; and the square of AC to the squares of AE, EC. For the same reason, the square of AF is equal to the squares of FC, CA, that is, to the squares of the segments EA, EB, EC, ED.

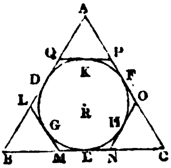
Cor. If AD, CB were joined, the sum of their squares would be equal to the sum of the squares of AC, DB; the sum of the squares of each pair being equal to the sum of the squares of the four segments EA, EC, EB, ED.

PROP. XVIII. PROB.—From a given equilateral and equiangular figure, to cut off as many triangles as it has sides, so that the remaining figure may be a regular polygon having twice as many sides as the given figure.

Let ABC be a given equilateral and equiangular figure: it is required to cut off from it as many triangles as it has sides, so that the remaining figure may be a regular polygon, having twice as many sides as the given one.

In the given figure inscribe (IV. A, cor.) the circle FDE, touching the sides in D, E, F: bisect the arcs DE, EF, FD in the points G, H, K; and through these draw tangents to the circle, cutting the sides in L, M, N, O, P, Q; QLMNOP is the polygon required.

For if straight lines be drawn from the centre R to D, E, F, the quadrilaterals ADRF, BDRE, CERF have each two right angles at the points of contact D, E, F: therefore (I. 32, cor. 1) the remaining angles A, DRF are equal to two right angles; and so likewise are the two B, DRE, and the two C, ERF. Take away the equal angles A, B, C, and the remaining angles DRF, DRE, ERF are equal: and therefore (III. 26) the arcs FD, DE, EF are equal. Hence also their halves, KD, DG, &c. are all equal, and (III. 29, and IV. 11, cor. 1) D, G, E, H, F, K are the angular points of a regular polygon inscribed in the circle, which has evidently twice as many sides as the given figure: and therefore (IV. B) LMNOPQ is the polygon required, since its sides touch the circle in the points G, E, H, &c.

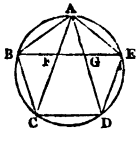


Schol. In case of an equilateral triangle ABC, it is plain that the construction will be effected simply by trisecting the sides, and joining the points of trisection LM, NO, PQ.

PROP. XIX. THEOR.—The diagonals of a regular pentagon which meet within the figure, divide one another in extreme and mean ratio: and the greater segment of each is equal to a side of the pentagon.

Let ABCDE be a regular pentagon, and AC, BE two of its diagonals which cut one another in F: AC, BE are each divided in extreme and mean ratio in F.

About the pentagon describe (IV. A, cor.) a circle. Then (III. 28) the arcs AB, BC, &c. are all equal, and AC is parallel to DE, and BE to CD: therefore CDEF is a parallelogram, and FC, FE are each equal to a side of the figure. Now, the triangles ABE, ABF have an angle common, and (III. 27) AEB is equal to BAF, because they stand on equal arcs: therefore (VI. 4, cor.) BE : BA :: BA : BF, or EB : EF :: EF : FB; whence (VI. def. 3) BE is divided in extreme and mean ratio in F: and it would be shown in the same manner that AC is also divided similarly in the same point.



Cor. The segment BG is divided similarly in F. For, in the similar triangles BFC, GFA, CF : FB :: AF : FG; or BG : BF :: BF : FG, because BG is equal to FC, and BF to AF.

PROP. XX. PROB.—On a given straight line to describe a regular pentagon, hexagon, or quindecagon.

In any circle (IV. 11, 15, or 16) inscribe a figure of the kind required: and (VI. 18) on the given line describe one similar to it.

Schol. It is plain, that in the same manner, on a given straight line, a regular polygon may be described similar to any of those the method of inscribing which in a circle was pointed out in the scholium to the 16th proposition of the fourth book; or, universally one similar to any regular polygon whatever, the method of inscribing which in a circle is known.

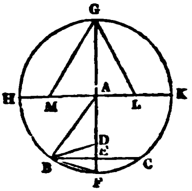
In practice, the construction may often be effected more easily without describing a similar figure in a circle. Thus, if it were required to describe a regular pentagon on CD (see the figure for the preceding proposition) CD may be divided in extreme and mean ratio, and on it the triangle CDA may be described having CA, DA, each equal to the sum of CD and its greater segment: then, if the triangles ABC, AED be described, having the sides AB, BC, AE, ED, each equal to CD, ABCDE will be the required figure, as is evident from the preceding proposition, and from the scholium to the eleventh proposition of the second book.

In like manner, to describe a regular decagon on a given straight line BF (see the figure for the next proposition) describe the triangle BAF, having the sides BA, FA each equal to the sum of BF and its greater segment, when it is cut in extreme and mean ratio: then A is the centre of the circle in which the required figure is to be inscribed, and the construction will be completed by inscribing nine other chords in the circle each equal to BF.

PROP. XXI. THEOR.—The square of the side of a regular pentagon inscribed in a circle, is equal to the squares of the sides of a regular hexagon and regular decagon, inscribed in the same circle.

Let AF, a radius of the circle BCG, be divided (II. 11) in D, so that the rectangle $AF.FD = AD^2$: inscribe the chord FB equal to AD: make the arc FC equal to FB, and join BC: then (IV. 11 and 15) BC, BF, AF are respectively the sides of the inscribed pentagon, decagon, and hexagon; and it is now to be proved that the square of BC is equal to the squares of BF, FA.

Produce FA to G, and join AB, BD. Then (by the demonstration of (IV. 10) AD, BD, BF are equal, and therefore, (III. 3, cor. 1) DE, EF are equal. Now (III. 35) the square of BE is equal to the rectangle GE.EF; and four times the square of BE, or (II. 4, cor. 2) the square of BC, is equal to the rectangle under GE, and four times EF; that is, since AF is equal to AG, and four times EF equal to twice DF, $BC^2 = 2AF.DF + 2AD.DF + 2DE.DF$. For 2AF.DF substitute $AD^2 + AD^2$, which (const.) is equal to it; and for 2DE.DF put its equal DF^2: then $BC^2 = AD^2 + AD^2 + 2AD.DF + DF^2$. But (II. 4) the last three terms are equal to the square of AF, and the square of AD to the square of BF: therefore $BC^2 = BF^2 + AF^2$.



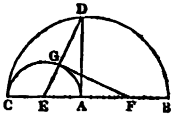
Cor. Hence we have the following easy and elegant method of inscribing a regular pentagon and decagon in a circle. Draw the two perpendicular diameters GF, HK: bisect AK, one of the radii, in L, and join GL: make LM equal to LG, and join GM. Then it is plain (II. 11) that the square of AM is equal to the rectangle AH.HM; and therefore AM is the side of the decagon; while MG is that of the pentagon, its square being (I. 47) equal to the squares of GA, AM, the sides of the hexagon and decagon.

PROP. XXII. PROB.—To divide a given straight line into two parts, such that the square of one of them may be equal to the rectangle contained by the other, and a given straight line.

Let AB, AC be two given straight lines: it is required to divide AB into two parts, such that the square of one of them may be equal to the rectangle under AC and the other.

On CB, the sum of the given lines, describe the semicircle CDB, and draw AD perpendicular to CB; bisect CA in E, and join DE; and make EF equal to ED: then the square of AF is equal to the rectangle AC.FB.

Describe the semicircle CGA, cutting ED in G, and join GF. Then, because FE, EG are respectively equal to DE, EA, and the angle FED common, GF is equal to AD, and the angle EGF, to EAD, which is a right angle, and therefore GF touches the circle CGA. Hence (III. 35) the rectangle CA.AB is equal to the square of AD, or of FG, or (III. 36) to the rectangle CF.FA. But (II. 1) the rectangle CA.AB is equal to the



two CA.AF, CA.FB, and (II. 3) the rectangle CF.FA is equal to CA.AF and the square of AF. From these equals take away the rectangle CA.AF, and there remains the rectangle CA.FB equal to the square of AF.

Schol. The eleventh proposition of the second book is the particular case of this problem in which the given lines are equal.

PROP. XXIII. PROB.—To divide a given straight line into two parts, such that the rectangle contained by the whole and one of the parts may have a given ratio to the square of the other.

Let AB (see the last figure) be the given line; it is required to divide AB into two parts, such that the square of one of them may have a given ratio to the rectangle under AB and the other.

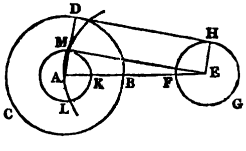
Let the given ratio of the rectangle to the square be that of AB to AC: then, the construction being made as in the last proposition, F is the point of section required. For, as was there proved, the rectangle CA.FB is equal to the square of AF. But (VI. 1) the rectangle CA.FB has to the rectangle AB.FB the same ratio that CA has to AB. Therefore the square of AF has to the rectangle AB.FB, the same ratio that CA has to AB.

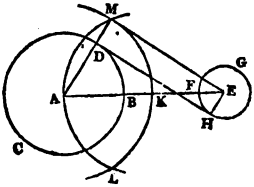
Schol. The eleventh proposition of the second book is also a case of this; being the same as this proposition, when the given ratio is a ratio of equality.

PROP. XXIV. PROB.—To draw a common tangent to two given circles.

Let BDC, FHG be given circles; it is required to draw a common tangent to them.

Join their centres A, E, and make BK equal to FE: from A as centre, with AK as radius, describe a circle cutting another, described on AE as diameter, in M: draw AM, meeting the circle BCD in D; and draw EH parallel to AM: join DH; it touches both the given circles.





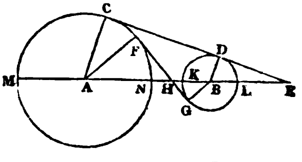
For MD, EH, which (const.) are parallel, are equal to one another, because each of them is equal to KB: therefore (I. 33) DH is parallel to ME. Now, (III. 31) the angle AME in a semicircle is a right angle; and therefore (I. 29) the angles ADH, DHE are right angles, and (III. 16, cor.) DH touches both the circles, since it is perpendicular to the radii AD, EH.

Schol. In the first figure the circles lie on the same side of the tangent, which is therefore exterior to them; but in the second the tangent is transverse, or lies between the circles. It is plain also, that in these figures, by using the point L instead of M, another exterior and another transverse tangent would be obtained: and this will always be so, when each of the circles lies wholly without the other, and does not touch it. But if the circles touch one another externally, the two transverse tangents coalesce into a single line passing through their point of contact: if they cut one another, there will be two exterior tangents, but no transverse one: if one of the circles touch the other internally, they can have only one common tangent, and this passes through their point of contact: and, lastly, if one of them lie wholly within the other without touching it, they can have no common tangent.

If the circles be equal, the points of contact of the exterior common tangents are the extremities of the diameters perpendicular to the line joining the centres: for (I. 33, and 29) the lines connecting the extremities of these towards the same parts are perpendicular to the diameters, and therefore (III. 16, cor.) they touch the circles.

PROP. XXV. THEOR.—If a common tangent to two circles cut the straight line passing through their centres, the segments between the point of intersection and the centres, are proportional to the radii.

Let A, B be the centres of two circles, and CD an exterior common tangent meeting AB produced in E; AE : BE :: AC : BD.



For the triangles CAE, DBE are equiangular, and therefore (VI. 4) AE : AC :: BE : BD, and alternately, AE : BE :: AC : BD.

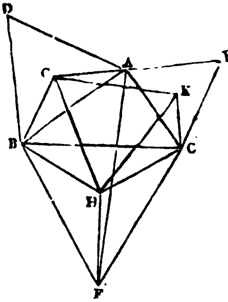
In like manner, if the transverse common tangent FG cut AB in H, the triangles AFH, BGH are equiangular, and therefore (VI. 4, and alternately) AH : HB :: AF : BG.

Cor. 1. Since AC : BD :: AE : BE, we have, by division, AC—BD : BD :: AB : BE. It would be shown, in a similar way, that the distance from B to the point in which the other exterior common tangent cuts AB produced would also be a fourth proportional to AC—BD, BD, and AB. The two exterior tangents, therefore, both pass through the same point E. It would appear, in like manner, by composition, from the analogy, AF : BG :: AH : HB, that the two transverse tangents both pass through the same point H.

Cor. 2. Since AE : BE :: AC : BD, and AH : HB :: AC : BD; therefore (V. 11) AE : BE :: AH : HB; and (VI. def. 5) AE is divided harmonically in H and B.

Prop. XXVI. Theor.—If on the three sides of any triangle, equilateral triangles be described, either all externally, or all internally; straight lines joining the centres of the circles inscribed in those three triangles, form an equilateral triangle.

On the three sides of any triangle ABC, let the equilateral triangles ABD, BCF, CAE be described externally, and find G, H, K, the centres of the circles described in those triangles; draw GH, HK, KG; GHK is an equilateral triangle.



Join GA, GB, HB, HC, HF, KC, AF. Then the angle FBC is two thirds of a right angle, and the angles GAB, GBA, FBH, BFH, each one third. The triangles FBH, ABG are therefore similar, and (VI. 4) FB : BH :: BA : BG; whence, alternately, FB : BA :: HB : BG; that is, in the triangles FBA, HBG, the sides about the angles FBA, HBG are proportional; and these angles are equal, each of them being equal to the sum of the angle ABC and two thirds of a right angle.[*] Hence (VI. 6) these triangles are equiangular; and therefore (VI. 4) FB or BC : FA :: BH : HG; and it would be shown in the same manner, by means of the triangles ACF, KCH, that FC or BC : FA :: CH : HK; therefore (V. 11) BH : HG :: CH : HK. But BH is equal to CH, and therefore (V. 9) HG is equal to HK; and it would be demonstrated in a similar manner, that HG, HK are each equal to GK.

If the equilateral triangles were described on the other sides of the lines AB, BC, CA, the angles ABF, GBH would be the difference between ABC and two thirds of a right angle; but the rest of the proof is the same.

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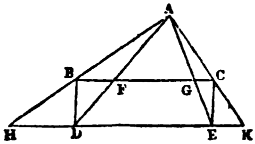
[*] If ABC exceed a right angle and a third, the sum of it and two thirds of a right angle is greater than two right angles. In that case, the angles ABF, GBH, understood in the ordinary sense, are each the difference between that sum and four right angles. If ABC be a right angle and a third, the sum becomes two right angles, and FB, BA are in the same straight line, as are also HB, BG. In this case it is proved as before, that FB : BA :: HB : BG; and then (V. E, cor.) FB or BC : FA :: HB : HG. The rest of the proof would proceed as above.

It may be remarked, that if an equilateral triangle be described on a straight line, and if on the two parts into which it is divided at any point, other equilateral triangles be described, lying in the opposite direction, the lines joining the centres of the three equilateral triangles will also form an equilateral triangle. The connexion of this and the proposition in the text will be perceived by supposing two angles of the triangle continually to diminish, till they vanish, as the triangle may thus be conceived to become a straight line.

PROP. XXVII. THEOR.—If on the hypotenuse of a right-angled triangle, a rectangle of any altitude be described, and straight lines be drawn from the right angle to the remote angles of the rectangle, dividing the hypotenuse into three parts; the rectangle of the extreme parts is to the square of the middle one, as the square of the altitude of the rectangle, is to the square of the hypotenuse.

On BC, the hypotenuse of the right-angled triangle ABC, let the rectangle BDEC be described, and let AD, AE be drawn cutting BC in F and G; $BF.GC : FG^2 :: BD^2 : BC^2$.

Produce AB, AC, and let them meet DE produced in H, K. Then (VI. 21) the triangles HDB, CEK are similar to one another, each of them being evidently similar to the triangle HAK. Therefore HD : DB :: EC or DB : EK; whence (VI. 17) $HD.EK = BD^2$. But it is easy to show that BF : GC :: HD : EK; and therefore, (VI. 1) $BF.GC : GC^2 :: HD.EK$ or $BD^2 : EK^2$; whence alternately, $BF.GC : BD^2 :: GC^2 : EK$. It is also plain, that GC : EK :: FG : DE or BC; and therefore (VI. 22) $GC^2 : EK^2 :: FG^2 : BC^2$. Hence (V. 11) $BF.GC : BD^2 :: FG^2 : BC^2$; and, alternately, $BF.GC : FG^2 :: BD^2 : BC^2$.



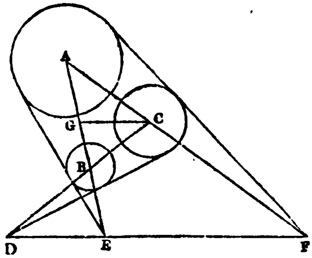
Schol. If, as in the figure for the 47th proposition of the first book of Euclid, BD, BC be equal, the rectangle BF.GC is equal to the square of FG; and therefore (VI. 17) FG is a mean proportional between BF and GC. In this case also it is manifest that if AB, AC be equal, the hypotenuse is trisected in F and G.

If, again, BD be the side of a square, of which BC is the diagonal, so that the square of BC is double of the square of BD, the square of FG is double of the rectangle BF.GC.

PROP. XXVIII. THEOR.—Let A, B, C be the centres of three unequal circles, and let the straight lines passing through them be cut in D, E, F by exterior common tangents to each pair: D, E, F are in the same straight line.

Join DE, EF, and draw CG parallel to FE: and for brevity put *a*, *b*, *c* to denote respectively the radii from the centres A, B, C. Then, (APP. I. 25) *c* : *a* :: CF : AF. But (VI. 2) CF : AF :: GE : AE; therefore (V. 11) *c* : *a* :: GE : AE. Also, (APP. I. 25) *a* : *b* :: AE : BE; therefore, *ex æquo*, *c* : *b* :: GE : BE. Again, *c* : *b* :: CD : BD; therefore (V. 11) GE : BE ::

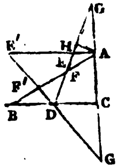
CD : BD; whence (VI. 2) GC, DE are parallel. Hence, since (const.) EF is parallel to GC, DE, EF are in the same straight



line; for if they were not, two intersecting straight lines would be both parallel to the same straight line, which (I. 29, cor. 2) is impossible.

PROP. XXIX. THEOR.—If through the point of bisection of the base of a triangle, a straight line be drawn cutting one of the other sides and the remaining one produced, and cutting a parallel to the base through the vertex; the straight line intercepted between the two extreme points thus obtained, is divided harmonically by the other two.

Let ABC be a triangle, D the middle point of its base, and AE a parallel to CB; draw any straight line through D, cutting AB, AE, CA produced, in F, E, G: then DG : GE :: DF : FE.



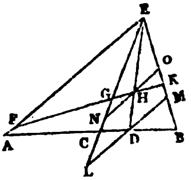
For, in the similar triangles GCD, GAE, GD : GE :: CD or BD : EA; and in the similar triangles BDF, AEF, BD : EA :: DF : FE. Hence (V. 11) GD : GE :: DF : FE; and therefore (VI. def. 5) DG is divided harmonically in F and E.

In the same manner it would be shown, that G′D : G′E′ :: DF′ : F′E′; and therefore E′G′ is divided harmonically in D and F.

Cor. Hence, if through D the middle point of the base, DFG be drawn, and if AE be drawn, so that GD is divided harmonically in E and F, AE is parallel to BC. For, if not, draw AH parallel to BC. Then, by this proposition, GH : HF :: GD : DF; and therefore (V. 11) GE : EF :: GH : HF, which (V. 14) is absurd. Therefore AE is parallel to BC.

PROP. XXX. THEOR.—If through the extremities and points of section of a straight line divided harmonically, straight lines be drawn to any point without the line, these four lines divide harmonically any straight line that falls upon them, or their continuations.

Let AB be a straight line divided harmonically in C and D, and E a point without it; join EA, EC, ED, EB: any straight line FGHK, falling on these, is divided harmonically in G and H.

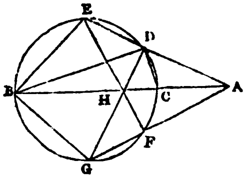


Through D draw LM (App. I. 7, cor. 2) so that LD may be equal to DM, and through H draw NO parallel to LM. Then, in the similar triangles LDE, NHE, and MDE, OHE, LD : NH :: DE : HE, and DM : HO :: DE : HE, whence (V. 11) LD : NH :: DM : HO; and therefore (V. 14) because LD, DM are equal, NH is equal to HO. Again, since LD, DM are equal, EF (App. I. 29, cor.) is parallel to LM, and therefore (I. 30) to NO: and since FGHK is drawn through the middle point of NO, the base of the triangle ENO, it is divided harmonically (App. I. 29) in G and H.

Prop. XXXI. Theor.—If through any point in the diameter of a circle produced, two straight lines be drawn making equal angles with the diameter, and cutting the circumference in four points, the straight lines joining the opposite points intersect one another in the diameter: and their point of intersection and the circumference divide the produced diameter harmonically.

Let A be a point in the diameter BC produced, and let the straight lines ADE, AFG, making equal angles with AB, cut the circumference in D, E, F, G: the straight lines joining DG, EF, cut BC in the same point H, and AB is divided harmonically in H and C.

Join BE, BG, DB, DC. Then (III. 8) AD is equal to AF, and AE to EG; and the angles ADG, AFE are equal, being supplements of the angles EDG, EFG in the same segment. Hence the triangles in which are these two angles and the equal angles at A, and which have the equal sides AD, AF, have (I. 26) their sides which are opposite to ADG, AFE, equal, and consequently DG, FE must cut AB in the same point H. Now, since AE, AG are equal, AB common, and the angles at A equal, the chords BE, BG are (I. 4) equal, and therefore (III. 28) the arcs themselves. Hence (III. 27) the angles BDE, BDG are equal, and BD bisects the exterior angle EDH of the triangle ADH. Again, BDC is (III. 31) a right angle, and is therefore equal to



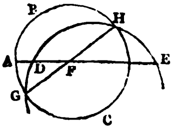
the two BDE, ADC: from these equals take the equals BDG, BDE, and the remaining angles CDH, CDA are equal: and therefore, since DC, DB bisect the interior and exterior angles at the vertex of the triangle ADH, BA is divided harmonically (VI. A, cor.) in C and H.

Schol. 1. Since BA : AC :: BH : HC, the ratio of BH to HC is always the same, so long as A is a fixed point: and therefore, however the positions of the points D, E, F, G may vary, provided the angles at A are equal, the transverse lines DG, FE will always cut the diameter in the same point H.

Schol. 2. Conversely, were a point H taken in the diameter, and transverse chords DG, EF drawn through it, making equal angles with the diameter, it would be shown in a similar manner, that the straight lines EDA, GFA, each passing through the extremities that lie on the same side of the diameter, would always meet in the same point in the continuation of the diameter, and that BA : CA :: BH : HC.

PROP. XXXII. THEOR.—If in the diameter of a circle and its continuation, two points be taken on the opposite sides of the centre, such that the rectangle under their distances from the centre may be equal to the square of the radius, any circle whatever described through these points, bisects the circumference of the other circle.

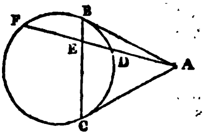
Let ABC be a circle, D a point in any diameter, and E a point in the same diameter produced on the opposite side of the centre F: then, if the rectangle DF.FE be equal to the square of the radius, any circle GDHE described through D, E, bisects the circumference of ABC.



Through G, one of the points of intersection of the circles, draw GFH a chord of GHE. Then (III. 35) the rectangle GF.FH is equal to the rectangle DF.FE, that is, (hyp.) to the square of the radius GF. Hence GF, FH are equal, and H is a point in the circumference of the circle ABC. But GH is drawn through its centre, and is therefore a diameter; wherefore its circumference is bisected in the points G, H.

PROP. XXXIII. THEOR.—A straight line drawn from the point of intersection of two tangents of a circle to the concave circumference, is divided harmonically by the convex circumference and the chord joining the points of contact.

Through the point A let the tangents AB, AC be drawn to the circle BFC, and let BC join the points of contact; any straight line ADEF drawn from A to the concave circumference, is divided harmonically in D and E.

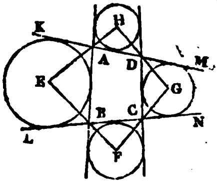


For (III. 17, schol.) AB, AC are equal; and therefore (II. 5, cor. 5) AB^2 or (III. 36) $FA.AD = AE^2 + BE.EC$; or (II. 1, and III. 35) $FE.AD + EA.AD = AE^2 + FE.ED =$ (II. 2) $AE.ED + EA.AD + FE.ED$. Take away EA.AD; then, since (II. 1) $AE.ED + FE.ED = AF.ED$, the remainder becomes $FE.AD = AF.ED$; whence (VI. 16) AF : AD :: EF : ED.

PROP. XXXIV. THEOR.—If four circles be described, either all outside, or all inside, of any quadrilateral, each of them touching three of its sides, produced or not produced; the circumference of a circle will pass through all their centres.

Let ABCD be the quadrilateral, and E, F, G, H the centres of the external circles, each touching three sides; a circle may be described through E, F, G, H.

Join AE, BE, BF, &c. Then, since AE, AH bisect two vertical opposite angles, HAD is equal to EAK, and (I. 14, cor.) EAH is a straight line. For the same reason, EBF, FCG, GDH are straight lines. Now (I. 13) the four exterior angles KAB, ABL, NCD, CDM, and the four interior ones, A, B, C, D, are equal to eight right angles; but the interior are equal to four right angles; therefore also the exterior are equal to the same. Again, the angles EAB, EBA are the halves of one pair of those exterior angles, and GCD, GDC the halves of the other pair: these four halves, therefore, are equal to two right angles; and if they be taken from four right angles, the sum of the angles of the triangles ABE, CDG, there remain the angles E, G equal to two right angles; and therefore (APP. I. 4) a circle may be described about the quadrilateral EFGH.



When the circles are described internally, the proof proceeds on the same general principle, but it is rather more simple.*

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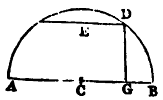
* If, in this case, the circle which touches three of the sides, touches also the fourth, the four circles, as also their centres, may be regarded as coinciding; and the circle passing through the four centres is reduced to a point.

PROP. XXXV. PROB.—To divide a given straight line into two parts, such that their rectangle may be equal to a given space, not exceeding (II. 5, cor. 2) the square of half the given line.

Let AB be a given straight line: it is required to divide it into two parts, such that their rectangle may be equal to a given space, not greater than the square of the half of AB.

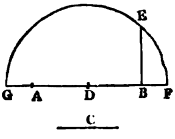


Bisect AB in C; then if the given space be equal to the square of AC or CB, AB is divided in C in the manner required. But if not, on AB describe a semicircle; draw ED parallel to AB, and at a distance equal to the side of a square containing the given space; and draw DG perpendicular to AB; G is the point of section required.

For (VI. 13) DG is a mean proportional between AG, GB, and (VI. 17) the rectangle AG.GB is equal to the square of GD; that is (const.) to the given space.

PROP. XXXVI. PROB.—To produce a given straight line, so that the rectangle under the whole produced line, and the part produced, may be equal to a given space.

Let it be required to produce a given straight line AB, so that the rectangle under the whole produced line, and the part added, may be equal to the square of a given straight line C.



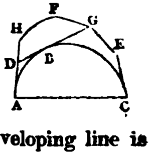
Bisect AB in D, and draw BE perpendicular to AB, and equal to C: from D as centre, at the distance DE, describe a semicircle cutting AB produced in F and G: either of these is the extremity of the produced part.

For (VI. 13) BE is a mean proportional between GB, BF, or AF, BF, since AF, GB are obviously equal. Therefore (VI. 17) the rectangle AF.FB is equal to the square of BE, that is, of C.

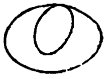
Schol. This proposition and the foregoing are the most useful cases of the 28th and 29th of the sixth book.

PROP. XXXVII. THEOR.—A line ABC, whether curved or polygonal, which is convex throughout, that is, which can be cut by a straight line in only two points, is less than any line, whether curved or polygonal, which envelops it, from one end to the other.

For, if ABC be not shorter than any of the lines that envelop it, some one of those lines will be less than any of the others, and will be either less than ABC, or, at most, equal to it. Let AHFGEC be that line, and draw the straight line DG cutting it in D, G, and either touching ABC, or not meeting it. Then DG is less than DHFG, and consequently (I. ax. 4) ADGEC is less than AHFGEC: but, by hypothesis, AHFGEC is less than any other enveloping line; and yet the enveloping line ADGEC is less than it, which is absurd; therefore every enveloping line is greater than ABC.



Schol. In exactly the same manner it may be shown, that a convex line returning into itself, is shorter than any line enveloping it on all sides, and either touching it, or encompassing it without touching it.

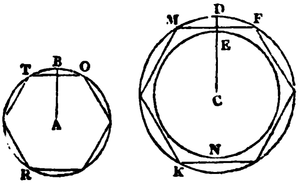


Prop. XXXVIII. Theor.—The circumferences of circles are to each other as their radii.

Let AB, CD be the radii of two circles; their circumferences are proportional to these.

For, if the circumference of the circle OR be not to that of FK, as AB to CD, AB will be to CD, as the circumference of OR to the circumference of a circle having a radius either less or greater than CD: and first, suppose its radius to be CE, less than CD. In the circle FK inscribe a regular polygon, the sides of which will not meet the circle EN, and in OR describe a similar polygon.*

Then, if AT, AO, CM, CF were joined, we should have (VI. 4) TO : MF :: AT or AB : CM or CD. But (VI. 20, cor. 4) the perimeter of the polygon in OR is to that of the polygon in FK, as TO to MF, that is, as AB to CD; and, by hypothesis, AB is to CD, as the circumference of OR to the circumference of EN; therefore (V. 11) the perimeter of the polygon in OR is to that of the polygon in FK, as the circumference of OR to the circumference of EN, which (V. 14) is absurd; as, by the preceding proposition, the first term of this analogy is less than the third, and the second greater than the fourth. Hence, AB cannot be to CD, as the circumference of OR to a circumference less than that of FK: and it would be proved in a similar manner, that CD cannot be to AB, as the circumference of FK to a circumference less than that of OR.



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* The method of effecting the construction here pointed out, is evident from the 16th proposition of the twelfth book.

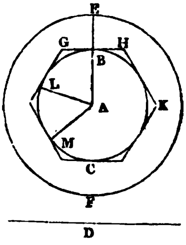
Neither can AB be to CD, as the circumference of OR to one greater than that of FK. For if it could, we should have, by inversion, CD to AB, as a circumference greater than that of FK to the circumference of OR, or, which is equivalent, as the circumference of FK to one less than that of OR. But this has been proved to be impossible: wherefore AB is to CD, as the circumference of OR to the circumference of FK.

Cor. Hence the arcs of sectors which have equal angles, are as the radii of the sectors. For the arcs are evidently proportional to the entire circumferences, the angles at the centres being equal.

PROP. XXXIX. THEOR.—The area of a circle is equal to the rectangle under its radius, and a straight line equal to half its circumference.

Let AB be the radius of the circle BC: the area of BC is equal to the rectangle under AB, and a straight line D equal to half the circumference.

For, if the rectangle AB.D be not equal to the circle BC, it is equal to a circle either greater or less than BC. First, suppose, if possible, the rectangle AB.D to be the area of a circle EF, of which the radius AE is greater than AB, and let GHK be a regular polygon described about the circle BC, such that its sides do not meet the circumference of EF. Then, by dividing this polygon into triangles by radii drawn to G, H, K, &c. it would be seen that its area is equal to the rectangle under AB and half its perimeter. But (APP. I. 37) the perimeter of the polygon is greater than the circumference of BC: and therefore the area of the polygon is greater than the rectangle AB.D, that is, by hypothesis, than the area of the circle EF, which is absurd, since the polygon is only a part of that circle, being contained within it. The rectangle AB.D, therefore, is not equal to the area of any circle greater than BC.



In the second place, this rectangle cannot be less than BC. For suppose, if possible, that it is equal to the area of a circle described from A as centre, within BC. Then, by describing about that circle a regular polygon, the sides of which would not meet BC, it would be shown as before, that the area of the polygon would be equal to the rectangle under half its perimeter and the radius of the interior circle, and would therefore be less than the rectangle AB.D, because AB is greater than the radius, and (APP. I. 37) D greater than half the perimeter. Therefore the polygon being less than AB.D, would be less than the circle about which it is described, which is absurd.

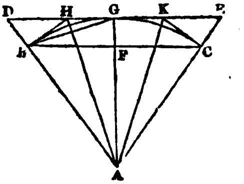
Hence the rectangle under the radius AB and half the circumference, is not equal to a circle greater than BC, nor to one less: it is therefore equal to BC.

Cor. 1. Hence the surface of a sector ALM is equal to the rectangle under its radius and half its arc. For the circle BC is to ALM, as the circumference of BC to the arc LM, sectors and arcs being proportional (VI. 33) to their angles at the centre. Hence, by halving the third and fourth terms, BC is to ALM, as D to ½ LM, or as D.AB to ½LM.AB: therefore (V. 14) ½LM.AB is equal to the sector ALM, because BC is equal to D.AB.

Cor. 2. Let the circumference of the circle whose diameter is unity, and consequently half the circumference of the circle whose radius is unity, be denoted by π. Then (App. I. 38) 1 is to π, as 2 AB to the circumference of BC. Hence we find the circumference to be $2\pi \times AB$; and multiplying half of this by the radius AB, we find the area to be equal to $\pi \times AB^2$. Hence *the area of any circle is found by multiplying the square of its radius by the constant number* π, which, as will be shown in the scholium to the next proposition, is 3·141593, nearly.

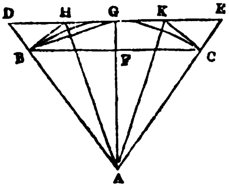
Prop. XL. Prob.—The area of a regular inscribed polygon, and that of a regular circumscribed one of the same number of sides being given; to find the areas of the regular inscribed and circumscribed polygons having double the number of sides.

Let A be the centre of the circle, BC a side of the inscribed polygon, and DE parallel to BC, a side of the circumscribed one. Draw the perpendicular AFG, and the tangents BH, CK. and join BG: then BG will be a side of the inscribed polygon of double the number of sides; and (IV. B) HK is a side of the similar circumscribed one. Now, as a like construction would be made at each of the remaining angles of the polygon, it will be sufficient to consider the part here represented, as the triangles connected with it are evidently to each other, as the polygons of which they are parts. For the sake of brevity, then, let P denote the polygon whose side is BC, and P′ that whose side is DE; and, in like manner, let Q and Q′, represent those whose sides are BG and HK: P and P′, therefore, are given; Q and Q′, required.



Now (VI. 1) the triangles ABF, ABG are proportional to their bases AF, AG, as they are also to the polygons P, Q; therefore AF : AG :: P : Q. The triangles ABG, ADG are likewise as their bases AB, AD, or (VI. 4) as AF, AG; and they are also as the polygons Q and P′: therefore AF : AG :: Q : P′; wherefore (V. 11.) P : Q :: Q : P′; so that Q is a mean proportional between the given polygons P, P′; and, representing them by numbers, we have $Q^2 = PP'$, so that *the area of* Q *will be computed by multiplying* P *by* P′, *and extracting the square root of the product*:

Again, because AH bisects the angle GAD, and because the triangles AHD, AHG are as their bases, we have GH : HD :: AG : AD, or AF : AB :: AHG : AHD. But we have already seen that AF : AG :: P : Q; and therefore (V. 11) AHG : AHD :: P : Q. Hence (V. E, cor.) AHG : ADG :: P : P+Q; whence, by doubling the antecedents, 2AHG : ADG :: 2P : P+Q. But it is evident, that whatever part the triangle ADG is of P′, the same part of the polygon Q′ is the triangle AHK, which is double of the triangle AHG. Hence the last analogy becomes Q′ : P′ :: 2P : P+Q. Now (V. Sup. 2, cor. 1) the product of the extremes is equal to the product of the means; and therefore Q′ *will be computed by dividing twice the product of* P *and* P′ *by* P+Q; and the mode of finding Q has been pointed out already.



Schol. This proposition enables us to find the approximate ratio of the diameter and circumference of a circle. Thus, let the radius be 1, and it is plain that the area of the circumscribed square is 4, and that of the inscribed one 2, being (I. 41) half of the circumscribed one. Put therefore P=2 and P′=4; and by the methods found above, we have the area of the inscribed octagon $Q=\sqrt{8}$, and the area of the circumscribed octagon $Q'=\frac{16^*}{2+\sqrt{8}}$: or by actual computation, Q=2·8284271, and Q′=3·3137085. By using these numbers instead of P and P′, we should find the areas of the inscribed and circumscribed regular polygons of sixteen sides. From them, in like manner, we should derive the areas of regular polygons of thirty-two sides: and thus, we may proceed as far as we please, each succeeding operation giving the areas of the inscribed and circumscribed polygons of double the number of sides of those employed in finding them. By this means the following table has been calculated, the decimals having been carried out to eight places, and the last rejected, so as to secure their accuracy for seven places.

Number of Sides.	Inscribed Polygon.	Circumscribed Polygon.
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
28	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025

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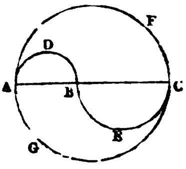
[*] By an easy and well-known algebraic operation (see the Author's Algebra, page 86), this may be reduced to the preferable form, $4(\sqrt{8}-2)$.

Number of Sides.	Inscribed Polygon.	Circumscribed Polygon.
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

Now, as the area of the circle is intermediate between the areas of the inscribed and circumscribed polygons, and as the areas of the last two in the table are the same in all the figures to which they are carried out, it follows that the area of the circle must also be 3·1415926: and as, by the last proposition, this is equal to the product of the radius 1, and half the circumference, the half circumference to the radius 1, or the whole circumference to the diameter 1, is the same number 3·1415926. This number is usually denoted by π.*

PROP. XLI. THEOR.—If the diameter of a circle be divided into any two parts, AB, BC, and if semicircles ADB, BEC be described on opposite sides of these, the circle is divided by their arcs into two figures GDE, FED, the boundary of each of which is equal to the circumference of FG; and which are such that AC : BC :: FG : FED, and AC : AB :: FG : GDE.



For (APP. I. 38) the circumferences of circles, and consequently the halves of their circumferences, are to one another as their diameters: therefore AB is to AC, as the arc ADB to AFC, and BC is to AC, as the arc BEC to AFC. Hence (V. 24) AC is to AC, as the compound arc ADEC to AFC: therefore ADEC is equal to half the circumference; and the entire boundaries of the figures GDE, FED are each equal to the circumference of FG.

Again, (XII. 2) circles, and consequently semicircles, are to one another as the squares of their diameters: therefore AC^2 is to AB^2 as the semicircle AFC to the semicircle ADB. Hence, since (II. 4) $AC^2 = AB^2 +$

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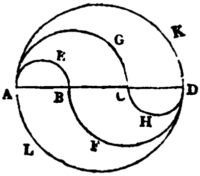
* This number, when carried out to more places, is found to be 3·14159265358979, &c.; and by the method now pointed out, or by others still easier, especially by series furnished by the modern analysis, it may be continued to any extent whatever. It cannot be determined, however, with complete precision; and hence the celebrated problem of the quadrature of the circle can be solved only by approximation.

The ratio of 1 to π is nearly that of 7 to 22, which gives by division 3·1428. This agrees with the value of π, so far as to produce an error in excess in the circumference, which is only a little more than the eight hundredth part of the radius: and on account of the smallness of the numbers, it is much used, when great accuracy is not required. The approximate ratio of 113 to 355, discovered by Metius, is also very convenient, and is so near the truth, as to give an error in excess in the circumference, not much exceeding a four millionth part of the radius.

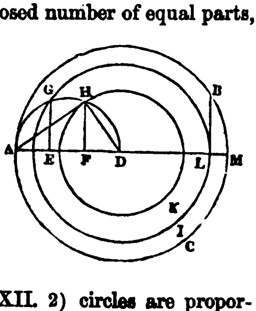
$2AB.BC+BC^2$, we find by conversion, that AC^2 is to $2AB.BC+BC^2$, as the semicircle AFC to the remaining space BDAFC; whence, by inversion, $2AB.BC+BC^2$ is to AC^2, as BDAFC to the semicircle AFC. But BC^2 is to AC^2, as the semicircle BEC to the semicircle AFC; and therefore (V. 24) $2AB.BC+2BC^2$ is to AC^2, as the compound figure FDE to the semicircle AFC. But (II. 3) $2AB.BC+2BC^2=2AC.BC$, and (VI. 1) $2AC.BC : AC^2 :: BC : \frac{1}{2}AC$. Hence the preceding analogy becomes BC to $\frac{1}{2}$AC, as FED to the semicircle, or by doubling the consequents, and by inversion, AC to BC, as FG to FED: and it would be proved, in the same manner, that AC : AB :: FG : GDE.

Cor. Hence we can solve the curious problem, in which it is required to divide a circle into any proposed number of parts, equal in area and boundary; as it is only necessary to divide the diameter into the proposed number of equal parts, and to describe the semicircles as in the annexed diagram. Then, whatever part AB is of AD the same part is LEF of the circle KL; and AC being double of AB, LGH is two of these parts: GEFH is therefore one of them; and so on, whatever is their number. Their boundaries are also equal, the boundary of each being equal to the circumference of the circle.*



Prop. XLII. Prob.—To divide a given circle ABC into any proposed number of equal parts, by means of concentric circles.

Divide the radius AD into the proposed number of equal parts, suppose three, in the points E, F, and through these points draw perpendiculars to AD, meeting a semicircle described on it as diameter in G, H; from D as centre, at the distances DG, DH, describe the circles GI, HK: their circumferences divide the circle into equal parts.



Join AH, DH. Then (VI. 20, cor. 2) AD, DH, DF being continual proportionals, AD is to DF, as a square described on AD is to one described on DH. But (XII. 2) circles are proportional to the squares of their diameters, and consequently to the squares of their radii. Hence (V. 11) AD is to FD as the circle

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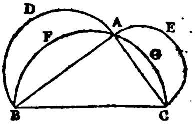
* Another solution would be obtained, if the circumference were divided into the proposed number of equal parts, and radii drawn to the points of division. This division, however, can be effected only in some particular cases by means of elementary geometry.

ABC to the circle HK; and therefore, since FD is a third of AD, HK is a third of ABC. It would be proved in a similar manner, that AD is to ED as ABC to GI. But ED is two thirds of AD, and therefore GI is two thirds of ABC; wherefore the space between the circumferences of GI, HK, is one third of ABC as is also the remaining space between the circumferences of ABC and GI.

Cor. Hence it is plain, that the area of any *annulus*, or ring, between the circumferences of two concentric circles, such as that between the circumferences of ABC and GI, is to the circle ABC as the difference of the squares of the radii DM, DL to the square of DM; or (II. 5, cor. 1, and III. 35) as the rectangle AL.LM, or the square of the perpendicular LB, to the square of DM: and it therefore follows (XII. 2) that the ring is equal to a circle described with a radius equal to LB.

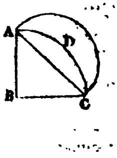
PROP. XLIII. THEOR.—If on BC, the hypotenuse of a right-angled triangle ABC, a semicircle BFAGC be described on the same side as the triangle, and if semicircles ADB, AEC be described on the legs, falling without the triangle, the *lunes* or *crescents*, ADBF, AECG, bounded by the arcs of the semicircles, are together equal to the right-angled triangle ABC.

For (XII. 2, and V. 15) the semicircle ADB is to BFGC, as the square of AB to the square of BC, and AEC to BFGC, as the square of AC to the square of BC; whence (V. 24) the two semicircles ADB, AEC taken together, have to BFGC, the same ratio, as the sum of the squares of AB, AC to the square of BC, that is, the ratio of equality. From these equals, take the segments AFB, AGC, and there remain the lunes DF, EG equal to the triangle ABC.



Cor. If the legs AB, AC be equal, the arcs AFB, AGC are equal, and each of them, an arc of a quadrant; also the radius drawn from A is perpendicular to BC: and since the halves of equals are equal, each lune is equal to half of the triangle ABC.

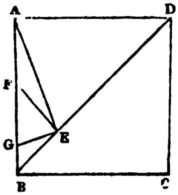
If, therefore, ABC be a quadrant, and on its chord a semicircle be described, the lune D comprehended between the circumferences is equal to the triangle ABC: and since (II. 14) a square can be found equal to ABC, we can thus effect the quadrature of a space D bounded by arcs of circles; though, as we have seen already (APP. I. 40, note) we cannot effect the quadrature of the circle itself, except by approximation.



Prop. XLIV. Theor.—The side and diagonal of a square are incommensurable to one another; that is, there is no line which is a measure of both.*

Let ABCD be a square, and BD one of its diagonals: AB, BD are incommensurable.

Cut off DE equal to DA, and join AE. Then, since (I. 5) the angle DEA is equal to the acute angle DAE, AEB is obtuse, and therefore (I. 19) in the triangle ABE, BE is less than AB, or than AD; wherefore AD is not a measure of BD. Draw EF perpendicular to BD. Then the angles FAE, FEA, being the complements of the equal angles DAE, DEA, are equal, and therefore AF, FE are equal. But (I. 32, cor. 4) ABD is half a right angle; as is also BFE, since BEF is a right angle: wherefore BE is equal to FE, and therefore to AF. From FB, which is evidently the diagonal of a square of which FE or EB is the side, cut off FG equal to FE, and join GE. Then it would be shown as before, that BG is less than BE; and therefore BE, the dif-



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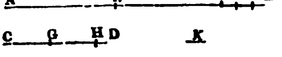
* To illustrate this proposition, let AB, CD be two magnitudes of which it is required to find a common measure. From AB the greater cut off as many parts as possible, AE, EF, each equal to CD, and let there be the remainder FB, which must be less than CD. From CD cut off parts CG, GH, each equal to FB, and let HD remain, which is less than FB. In like manner from FB cut off parts each equal to HD, and suppose that HD is contained without remainder in FB: then HD is a common measure of AB and CD. For, since HD measures FB, it measures also CH, which is a multiple of FB; and it must also measure the sum of CH and HD, that is, CD, which is one of the proposed lines. But, since HD is a measure of CD, it is a measure of AF, which is a multiple of CD; and being also a measure of FB, it is a measure of the sum of AF, FB, that is, of AB: it is therefore a common measure of AB, CD.



Now, any measure whatever of AB, CD is either HD, or a submultiple of it. For, let K be a measure of AB, CD; then K is also a measure of AF, which is a multiple of CD; and being a measure of AB, it is a measure of FB, the difference of AB, AF. Hence, since K is a measure of FB, it is a measure of CH, which is a multiple of FB; and being a measure of CD and CH, it must also be contained without remainder in their difference HD; therefore K is either equal to HD, or is a submultiple of it: and hence it follows, that HD is the *greatest* common measure of AB, CD, every other being a submultiple of HD. It is also plain, that if, by the continuation of processes such as that employed in finding HD, no magnitude can be found which will be contained without remainder in the one preceding it, the magnitudes have no common measure, or are incommensurable, as is the case with regard to the side and diagonal of a square. This subject will receive farther illustration from attempting to express numerically (I. 47, cor. 1) the length of the diagonal of a square; as it will be found, that if the side be expressed by a whole number, the diagonal cannot be exactly expressed by any number whatever, but may be approximated as nearly as we please, by means of a decimal or other fraction.

ference between the side and diagonal of the square AC, is contained twice in the side AB, with the remainder GB, which is itself the difference between the side FE or EB, and the diagonal FB of another square. By repeating the process, we should find, in exactly the same manner, that BG would be contained twice in BE, with a remainder, which would be the difference between the side and diagonal of a square described on BG: and it is evident that a like process might be repeated continually, as no excess of a diagonal above a side would be contained in the side without remainder: and as this process has no termination, there is no line, however small, which will be contained without remainder in both AB and BD ; they are, therefore incommensurable.

BOOK II.

DEFINITIONS.

1. A LINE which is such, that any point whatever in it fulfils certain conditions is called the *locus* of that point.

2. A *porism* is a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions.

3. *Isoperimetrical* figures are such as have their perimeters, or bounding lines, equal.

Schol. 1. Several instances of loci have already occurred in the preceding books.

1. Thus, it was proved in the 37th proposition of the first book, that all triangles on the same base, and between the same parallels, are equal in area: and hence, if only the base and area of a triangle be given, its vertex may be at any point in a straight line parallel to the base, and at a distance from it which may be determined by applying (I. 45, cor.) to half the base a parallelogram equal to the given area; and therefore the parallel is the locus of the vertex. Here the conditions fulfilled are, that straight lines drawn from any point in the parallel to the extremities of the given line, form with it a triangle having a given area.

2. It was proved in the 21st proposition of the third book, that all angles in the same segment of a circle are equal: and hence, if only the base and vertical angle of a triangle be given, the vertex may be at any point of the arc of a segment described on the base, in the manner pointed out in the 33d proposition of the third book: that arc, therefore, is the locus of the vertex.

3. It was shown in proposition G of the sixth book, that straight lines drawn from any point whatever in the circumference of the circle ABGC to the points E, F, have the same ratio — that of EA to AF. Hence, therefore, if the base of a triangle, and the ratio of the sides be given, the locus of the vertex is the circumference of the circle described in the manner pointed out in the corollary to that proposition; unless the ratio be that of equality, in which case the locus is evidently a perpendicular bisecting the straight line joining the points.

4. It follows likewise, from the fourth corollary to the 47th proposition of the first book, that when the base of a triangle and the difference of the squares of the sides are given, if the point D be found* in the base BC, or its continuation, such that the difference of the squares of BD, CD is equal to the difference of

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* The point D may be determined in the manner pointed out in the first note in page 54.

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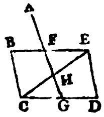
the squares of the sides; and if through D a perpendicular be drawn to BC, straight lines drawn from any point of the perpendicular to B, C, will have the difference of their squares equal to the given difference; and hence the perpendicular is the locus of the vertex, when the base and the difference of the squares of the sides are given.

5. It will appear, in a similar manner, from proposition A of the second book, that if BC the base of a triangle, and the sum of the squares of the other sides AB, AC, be given, the locus of the vertex is the circumference of a circle described from D, the middle point of the base as centre, and with the radius DA. To find DA, take the diagonal of the square of BD as one leg of a right-angled triangle, and for its hypotenuse take the side of a square equal to the given sum of the squares of AB, AC: then the diagonal of the square described on half the remaining leg of that right-angled triangle will be the radius DA. The proof of this is easy, depending on the second and third corollaries to the 47th proposition of the first book, and on A of the second book.

Schol. 2. Porisms may be regarded as having their origin in the solution of problems, which, in particular cases, on account of peculiar relations in their data, admit of innumerable solutions; and the proposition announcing the property or relation which renders the problem indeterminate, is called a porism. This will be illustrated by the solution of the following easy problem.

Through a given point A, let it be required to draw a straight line bisecting a given parallelogram BCDE.

Suppose AFG to be the required line, and let it cut the sides BE, CD in F, G, and the diagonal CE in H. Then from the equal figures EBC, FBCG take FBCH, and the remaining triangles EHF, CHG are equal. Now, since (I. 29 and 15) these triangles are equiangular, it is evident that they can be equal in area only when their sides are equal: wherefore H is the middle point of the diagonal. The construction, therefore, is effected by bisecting the diagonal EC in H, and drawing AFHG. For the triangles CHG, EHF are equiangular, and since CH, HE are equal, the triangles are equal. To each of them add the figure FBCH; then the figure FBCG is equal to the triangle EBC, that is, to half the parallelogram BD.



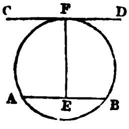
Now, since the diagonal CE is given in magnitude and position, its middle point H is given in position, and therefore H is always a point in the required line, wherever A is taken. Hence, so long as A and H are different points, the straight line AHG (I. post. 1) is determined. This, however, is no longer so, if the given point A be the intersection of the diagonals, that is, the point H, as in that case only one point of the required line is known, and the problem becomes indeterminate, any straight line whatever, through H, equally answering the conditions of the problem: and we are thus led by the solution of the problem to the conclusion, that *in a parallelogram a point may be found, such that any straight line whatever, drawn through it, bisects the parallelogram*; and this is a porism.

The 32d proposition of the preceding book, when considered in a particular manner, affords another instance of a porism; as it appears that if a circle and a point D or E be given, another point E or D may be found, such that any circle whatever, described through D and E will bisect the circumference of the given circle; and this may be regarded as the indeterminate case of the problem, in which it is required, through two given points to describe a circle bisecting the circumference of another given circle;—a problem which is always determinate, except when the points are situated in the manner supposed in the proposition referred to.

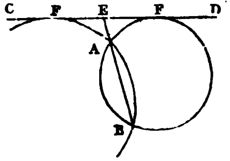
PROPOSITIONS.

PROP. 1. PROB.*—Through two given points, to describe a circle touching a straight line given in position.

Through two given points A, B, let it be required to describe a circle, touching a given straight line CD.



Join AB. Then, if CD be parallel to AB, bisect AB by the perpendicular EF, and (III. 1, schol.) through A, F, B describe a circle: it is the circle required. For (III. 1, cor.) EF passes through the centre, and (I. 29) is perpendicular to CD; therefore (III. 16) the circle touches CD; and it passes through A, B.



But if CD, BA be not parallel, let them meet in E: between AE, BE find a mean proportional, and make EF equal to it: through B, A, F describe a circle: it is the one required. For (const. and VI. 17) the rectangle BE.EA is equal to the square of EF, and therefore (III. 37) EF touches the circle; and the circle passes through the given points.

Schol. 1. In this latter case, since EF may be taken on either side of E, there are two circles that answer the conditions of the problem; and these are evidently unequal, unless BAE be perpendicular to CD.

Schol. 2. If one of the points A be in the given line, the pre-

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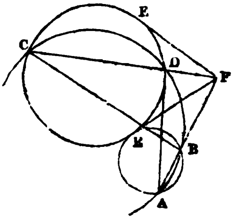
[*] This is one of the problems known under the name of *tangencies*. The general problem of the tangencies, as understood by the ancients, is as follows: Of three points, three straight lines, and three circles of given radii, any three being given in position; it is required to describe a circle passing through the points, and touching the straight lines and circles. This general problem comprehends ten subordinate ones, the data of which are as follows: (1.) three points; (2.) two points and a straight line; (3.) two points and a circle; (4.) a point and two straight lines; (5.) a point, a straight line, and a circle; (6.) a point and two circles; (7.) three straight lines; (8.) two straight lines and a circle; (9.) a straight line and two circles; and (10.) three circles. The first and seventh of these are the fifth and fourth propositions of the fourth book of Euclid: the rest are given here.

If a circle be continually diminished, it may be regarded as becoming ultimately ceding solution fails, as it does not give a third point in the required circumference. In that case two straight lines, one bisecting AB perpendicularly, and the other drawn through A perpendicular to CD, will intersect each other in the centre of the required circle. The proof is plain from the fourth proposition of the first book, and the sixteenth of the third book.

PROP. II. PROB.—Through two given points, to describe a circle touching a given circle.

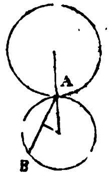
Through the given points A, B, let it be required to describe a circle touching a given circle CDE.

Join AB. Then, if A, B be equally distant from the centre of the given circle, bisect AB by a perpendicular, cutting the given circle in two points, and it is plain that a circle described through either of these points, and through A, B, will touch the given circle.



But if A, B be not equally distant from the centre, take any point C in the circumference of CDE, and describe a circle through A, B, C: if this touch the given circle, it is a circle such as is required. But if it do not, let it cut CDE again in D, and draw CDF meeting AB produced in F: draw (III. 17) FE touching the given circle in E: through ABE describe a circle: it is the one required.

Join AD, BC. Then (III. 21) the angles A, C are equal, as they stand on the same arc BD: also AFC is common to the two triangles AFD, CFB. Therefore (VI. 4, cor.) AF : FD : : CF : FB; and hence (VI. 16) the rectangles AF.FB, CF.FD are equal. But (III. 36) CF.FD is equal to the square of EF: wherefore also AF.FB is equal to the square of EF. ABE therefore (III. 37) touches the straight line FE; and consequently it touches the circle CDE.



Schol. If one of the points A be on the given circumference, the preceding method fails, as it does not give a third point in the required circumference. In that case the construction is effected simply by drawing a radius to A, and producing it, if necessary, to meet a perpendicular bisecting AB: their point of intersection is evidently the centre of the required circle.

In this case there is but one circle that answers the conditions of the problem. When the points are both without the circle, or both within it, there are two circles; but if one be without and the other within, the problem is manifestly impossible.

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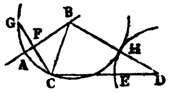
a point. By being continually enlarged, on the contrary, it may have its curvature so much diminished, that any portion of its circumference may be made to differ

Cor. 1. Hence in a given straight line AB, a point may be found, such that the difference of its distances from two given points C, D, may be equal to a given straight line.



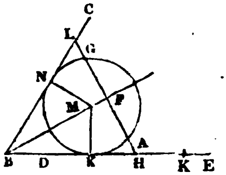
Join CD, and from D as centre, with a radius, DE equal to the given difference, describe a circle; draw CFG perpendicular to AB, and make FG equal to FC; through C, G, describe a circle touching the other circle; join B, D, the centres of the two circles, and draw BC: BC, BD are evidently the required lines.

Cor. 2. In the same manner, if a circle were described from D as centre, with the sum of two given lines as radius, and a circle were described through C and G touching that circle, straight lines drawn from the centre of that circle to C and D, would be equal to the given sum.

PROP. III. PROB.—Through a given point, to describe a circle touching two given straight lines.

Through a given point A, let it be required to describe a circle touching two given straight lines, BC, DE.

First, let BC, DE not be parallel, but let them meet in B: draw BF bisecting the angle CBE: and through A, draw LFAH, perpendicular to BF, cutting DE in H, and BC in L; and make FG equal to FA. Take HK a mean proportional between AH, HG; through K, A, G describe a circle: it is the one required.



For, in the triangles BFL, BFH, the angles at B and F are equal, and BF common: therefore (I. 26) FL, FH are equal; and (I. ax. 3) LG, AH are equal. But (const. and VI. 17) the rectangle GH.HA is equal to the square of HK; and therefore (III. 37) DE touches the circle. Now (III. 1, cor.) since FA, FG are equal, and FB perpendicular to AG, the centre of the circle KAG is in FB: let it be M, and join MK; draw also MN perpendicular to BC. Then (I. 26) in the triangles MBK, MBN, MN is equal to MK, and the circle will pass through N, and (III. 16) will touch BC, since BNM is a right angle.

Secondly, let BC, DE be parallel, and A be between them.

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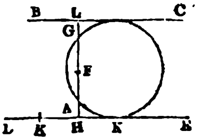
in as small a degree as we please from a straight line. Viewing the subject in this light, we may regard the first nine of the problems, now mentioned, as comprehended in the tenth. Thus, we shall have the first, by supposing the circles to become infinitely small; the seventh, by supposing them infinitely great; the fifth, by taking one of them infinitely small, one infinitely great, and one as a circle of finite magnitude: and so on with regard to the others. These views of the subject tend to illustrate it; but they do not assist in the solution of the problems.

Through A draw LAH perpendicular to the parallels: bisect HL in F, and make FG equal to FA: take HK a mean proportional between GH, HA; and a circle described through GAK is the one required. The proof is easily derived from that of the preceding case.



Schol. If the given point A be in one of the given lines DE, the foregoing solution fails. In that case, find F as before, and a perpendicular through A to DE will intersect BF in the centre of the required circle.

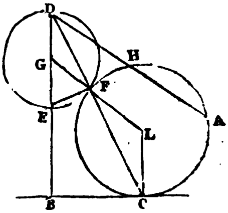
In this case, there may be two circles, one touching DE, BC, and the other DE and the continuation of BC, through B, unless BC, DE be parallel, when there is but one circle. When A is not on BC or DE, there may obviously be two circles, since HK may be taken on either side of H.

PROP. IV. PROB.—Through a given point, to describe a circle, touching a given straight line, and a given circle.

Through a given point A, let it be required to describe a circle touching a given straight line BC, and a given circle DEF.

Through G, the centre of DEF, draw DEB perpendicular to BC, and join AD; to AD, DB, DE, take DH a fourth proportional; through A, H, describe (App. II. 1) the circle HAC touching BC; HAC is the circle required.

Join the centre L, and the point of contact C: then (III. 18) LCB is a right angle, and LC, DB are parallel. Draw CD, cutting the circle DEF in F, and join EF. Then the triangles DBC, DFE are similar, and therefore CD : DB : : DE : DF; whence (VI. 16) the rectangle CD.DF is equal to BD.DE: but (const. and VI. 16) AD.DH is equal to BD.DE: therefore CD.DF is equal to AD.DH. Hence (App. I. 5, schol.) the point F is also in the circumference of AHC. Join FG, FL. Then (I. 29) the angles GDF, LCF are equal; wherefore DFG, CFL, which (I. 5) are respectively equal to these, are equal to one another: and DFC being a straight line, GFL (I. 14, cor.) is also a straight line. Hence (App. I. 2, cor.) the circles touch one another in F; for they meet in F, and the straight line joining their centres passes through that point.



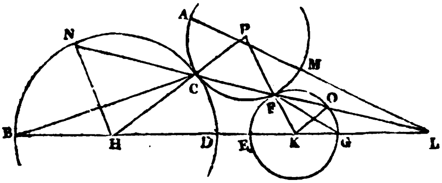
Schol. This solution fails, if A be either at C, any point in BC, or at F, any point in the circumference of DEF. The construction is then effected simply by drawing DFC, GFL, and the perpendicular CL; as CL cuts GL in the centre of the required circle.

It may be farther remarked, that, when the given point, straight line, and circle, are situated as in the diagram here employed, there will be two circles obtained by the construction given above, each of which will be touched externally by the given circle. Besides these, two others, which will be touched internally by the given circle, will be obtained, by joining AE instead of AD, and making the variations thence arising. In making the construction in this manner, H is taken in the continuation of the line joining A with the extremity of the diameter. The variations in the proof present no difficulty.[*]

PROP. V. PROB.—Through a given point, to describe a circle touching two given circles.

Through a given point A, let it be required to describe a circle touching two given circles, BCD, EFG.

Through the centres H, K, draw BDEGL, and (VI. 10, schol.) find the point L, such that HL is to KL, as the radius of the greater circle to the radius of the less : join AL, and (VI. 12) take ML such that AL : BL :: GL : LM: then (APP. II. 2) through A and M describe the circle ACM, touching BCD: it is the circle required.



Through the point of contact C, draw LCN: join HN, HC, and let KF, KO be drawn parallel to them, and meet LN in F, O. Then (VI. 4 and alternately) HL : KL :: HN : KF, and HL : KL :: HC : KO; but (const.) HL is to KL, as the radius of BCD to the radius of EFG; therefore KF, KO are radii of EFG, and LN cuts EFG in F and O. Join BC, FG. Then the triangles LBC, LFG have an angle common, and the angles LBC, LFG equal, because (III. 20) they are halves of the equal angles DHC, GKO; and therefore BL : LC :: FL : LG, and (VI. 16) the rectangle BL.LG is equal to CL.LF. But (const. and VI. 16) BL.LG is equal to AL.LM: therefore the rectangles CL.LF, AL.LM are equal, and (App. I. 5, cor.) F is in the circumference of ACM. Produce HC, KF to meet in P. Then the angles PCF, PFC are equal, being equal to HCN, KFO, which are each equal to HNC: PC therefore is equal to PF. Now HP passes through the centre of ACM, for (III. 12) the straight line joining H with the centre of ACM passes through C. Hence, since CP, PF are equal, P is evidently the centre;* and therefore (App. I. 2, cor.) ACM touches EFG in F.

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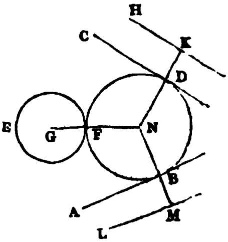
[*] In this problem, the straight line may fall without the circle, may cut it, or may touch it: the point may be without the circle, within it, or in its circumference; or it may be in the given straight line, or on either side of it: and it will be an interesting exercise for the student, in this and many other problems, to consider the variations arising in the solution, from such changes in the relations of the data, and to determine what relations make the solution possible, and what render it impossible. It may also be remarked, that in many problems, there will be slight variations in the proofs of different solutions of the same problem, even when there is no change in the method of solution; such as in the present instance, when the required circle is touched externally, and when internally. Thus, while in one case angles may coincide, in another the corresponding ones may be vertically opposite; and the reference may sometimes be to the third, and sometimes to the fourth proposition of the first book of this Appendix. It is, in general, unnecessary to point out these variations, as, though they merit the attention of the student, they occasion no difficulty.

Prop. VI. Prob.—To describe a circle touching two given straight lines and a given circle.

Let AB, CD be two straight lines given in position, and EF a given circle: it is required to describe a circle touching AB, CD, EF.

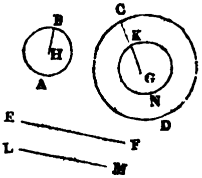


Draw HK, LM parallel to CD, AB, at distances from them equal each to the radius GF: find (App. II. 3) N, the centre of a circle passing through G, and touching HK, LM: draw GFN, and from N as centre, with NF as radius, describe the circle FDB: it is the circle required.

Draw the perpendiculars NK, NM: these are each equal to GN; and GF, KD, MB being equal, NF, ND, NB are also equal, and the circle must pass through B, D: it also touches AB, CD, because the angles at B and D are right angles.

Prop. VII. Prob.—To describe a circle touching two given circles, and a given straight line.

Let AB, CD be the given circles, and EF the given straight line: it is required to describe a circle touching AB, CD, EF.



Draw LM parallel to EF, and at a distance from it equal to BH; draw the radii CG, BH, and make CK equal to BH; with GK as radius, and G as centre, describe a circle KN; find (App. II. 4) the centre of a circle touching KN and LM, and passing through H: it would be shown in nearly the same manner as in the preceding proposition, that this is also the centre of the required circle.

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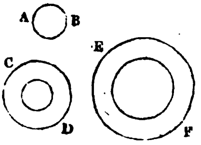
* This is easily proved indirectly, by means of the seventh proposition of the third book.

Prop. VIII. Prob.—To describe a circle touching three given circles.

Let AB, CD, EF be three given circles: it is required to describe a circle touching them.

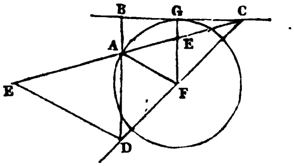
From the centre of CD, with a radius equal to the difference or sum of the radii of AB and CD, describe a circle; and from the centre of EF, with a radius equal to the difference or sum of the radii of AB and EF, describe a circle: then (App. II. 5) find the centre of a circle touching the circles now described, and passing through the centre of AB: it will also be the centre of the required circle. The proof is obvious.



Prop. IX. Prob.—Through a given point A, to describe a circle, touching one given straight line BC, and having its centre in another DC.

If the lines be parallel, the radius will evidently be their perpendicular distance asunder, and the construction is obvious.

But, if they be not parallel, let them meet in C, and draw a straight line through C and A; through A draw BD perpendicular to BC; from D as centre, at the distance DB describe a circle cutting CA in E; join ED; and through A draw AF parallel to ED meeting CD in F: F is the centre of the required circle.



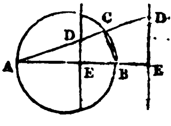
Draw FG perpendicular to BC. Then, because the triangles CDE, CFA, and CDB, CFG are similar, ED : AF :: DC : FC, and BD : GF :: DC : FC; whence ED : AF :: BD : GF; and (V. 14) AF, GF are equal, because ED is equal to BD. A circle therefore described from F as centre, with FG as radius, passes through A; and it touches BC, because the angles at G are right angles.

Schol. It is plain, that with the relations of the data in the diagram here employed, there may be two solutions, as either of the points E may be used. Should the circle described with DB as radius, merely touch AC, there would be but one solution: should it not meet CA, the solution would be impossible.

Prop. X. Theor.—If a chord of a given circle have one extremity given in position, and if a segment terminated at that extremity, be taken on the chord, produced if necessary, such that the rectangle under the segment and chord may be equal to a given space; the locus of the point of section is a straight line given in position.

Through a given point A in the circumference of the given circle ABC, let a chord AC be drawn, and in AC or its continuation, let D be taken, so that the rectangle AC.AD may be equal to a given space: the point D is always in a straight line given in position, however the point C may vary its position on the circumference.

Draw the diameter AB, and to AB and the side of a square equal to the given space, take AE a third proportional, so that the rectangle AB.AE may be equal to the given space; through E draw DE perpendicular to AB: DE is the locus of D.



Join BC. The triangles ABC, ADE have the angle A common, and the angles ACB, AED right angles: therefore AB : AC :: AD : AE; and hence (VI. 16) the rectangle AC.AD is equal to AB.AE, that is (const.) to the given space. In the same manner, it may be shown, that wherever D is taken in the straight line DE, the rectangle AC.AD is equal to the given space. Therefore (App. II. def. 1) DE is the locus of D; and its position is given from that of the point E.*

Schol. If, on the contrary, the point A and the straight line DE were given, and if, in a revolving straight line AD, or its continuation, the point C were taken, so that the rectangle DA.AC should be always equal to the same given space; it would be shown in a similar manner, that the locus of C would be a given circle ACB.

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* In discovering loci, as well as in other investigations in geometry, the student is assisted by what is termed *geometrical analysis;* of the nature of which it may be proper here to give some explanation.

If, in the foregoing proposition, instead of being informed that the locus is a straight line, we were required to find what the locus is, we might proceed in the following manner: Let D be any point in the required line, so that the rectangle AC.AD is equal to the given space; and having drawn the diameter AB, find E so that the rectangle AB.AE may be equal to AC.AD, and therefore E a point in the required line; and join DE, BC. Then (VI. 16) AB : AC :: AD : AE. Hence (VI. 6) the triangles DAE, BAC, having the angle A common, are equiangular; and therefore AED is equal to ACB, which is a right angle. The point D is therefore in a perpendicular passing through E; and in the same manner it would be shown, that any other point in the required line is in the perpendicular; that is, the perpendicular is its locus.

The investigation now given is called the *analysis* of the proposition, while the solution in the text is called the *synthesis* or *composition*. In analysis we commence by supposing that to be effected, which is to be done, or that to be true, which is to be proved; and by a regular succession of consequences founded on that supposition, and on one another, we arrive at something which is known to be true, or which we know the means of effecting. The synthesis then commences with the conclusion of the analysis, and retraces its several steps, making that precede which before followed, till we arrive at the required conclusion. From this it appears that analysis is the instrument of investigation; while synthesis affords the means of communicating what is already known; and hence, in the Elements of Euclid, the synthetic method is followed throughout. What is now said will receive farther illustration from the solution of the following easy problem.

Given the perimeter and angles of a triangle, to construct it.

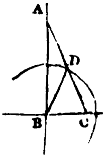
Analysis.—Suppose ABC to be the required triangle, and produce BC both ways, making BD equal to BA, and CE to CA: then DE is given, for it is equal to the

Prop. XI. Theor.—The locus of the middle point of a straight line given in magnitude, and subtending a right angle given in position, is the circumference of a circle given in magnitude and position.

Let ABC be a right angle given in position, and let there be a straight line of given length subtending it: then, while that straight line changes its position in such a manner that its extremities continue always on AB, BC, its point of bisection, describes the circumference of a circle.

From B as centre, with the half of the given straight line as radius, describe a circle: it is the locus required.

Draw BD, any radius of that circle: from D as centre, with DB as radius, describe an arc cutting BA in A, and draw ADC. Then the angles BAD, ABD are equal; and (I. ax. 3) their complements DBC, DCB are equal. Hence DC is equal to DB, and therefore to DA; and AC is equal to the given line, since DB is equal to its half. Its point of bisection, also, is on the circumference, since AD, DC are equal.*



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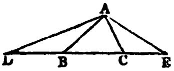
sum of the three sides AB, BC, CA; that is, it is equal to the given perimeter. Join AD, AE. Then (I. 5) the angles D and DAB are equal, and therefore, each of them is half of ABC, because (I. 32) ABC is equal to both. The angle D therefore is given; and in the same manner it may be shown that E is given, being half of ACB. Hence the triangle ADE is given, because the base DE, and the angles D, E are given: and ADE being given, ABC is also given, the angle DAB being equal to D, and EAC equal to E.



Composition.—Make DE equal to the given perimeter, the angle D equal to the half of one of the given angles, and E equal to the half of another; draw AB, AC, making the angle DAB equal to D, and EAC to E, ABC is the triangle required.

For (I. 6) AB is equal to BD, and AC to CE. To these add BC, and the three AB, BC, CA are equal to DE, that is to the given perimeter. Also (I. 32) the angle ABC is equal to D and DAB, and is therefore double of D, since D and DAB are equal. But D is equal to the half of one of the given angles; therefore, ABC is equal to that angle: and, in the same manner, ACB may be proved to be equal to another of the given angles. ABC therefore is the triangle required, since it has its perimeter equal to the given perimeter, and its angles equal to the given angles.

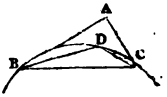
It is impossible to give rules for effecting analyses, that will answer in all cases. It may be stated, however, in a general way, that when sums or differences are concerned, the corresponding sums or differences should be exhibited in the assumed figure; that in many cases remarkable points should be joined; or that through them lines may be drawn perpendicular or parallel to remarkable lines, or making given angles with them: and that circles may be described with certain radii, and from certain points as centres; or touching certain lines; or passing through certain points. Some instances of analysis will be given in subsequent notes: and the student will find it useful to make analyses of many other propositions, such as several in this Appendix, both in the first book, and in the others.

* The analysis of this proposition is easily obtained from proposition A of the second book. For, since $AB^2 + BC^2$, or AC^2, or (II. 4, cor. 2) $4CD^2 = 2CD^2 + 2BD^2$; take away $2CD^2$, and halve the remainder: then $CD^2 = BD^2$; and therefore $BD = CD$, half the given line. Hence the distance from the point of bisection D to the point B is equal to half the given line. D, therefore, is always in the circumference of a circle described from B as centre, with half the given line as radius.

Schol. If the extremity C pass to the other side of the centre B, the middle point will describe one quadrant, while the extremity A descends to B, another while it continues to descend below B, and a third while it re-ascends to B.

Prop. XII. Theor.—If the base of a triangle be given in magnitude and position, and the vertical angle in magnitude, the locus of the centre of the inscribed circle is a given circle.

Let ABC be a triangle of which the base BC is given in magnitude and position, and the angle A in magnitude; the locus of D, the centre of the inscribed circle, is a given circle.

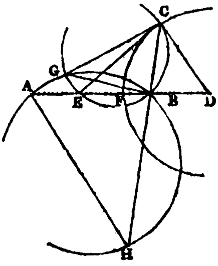


Join DB, DC: these (IV. 4) bisect the angles at B and C. Then, by drawing a straight line through A, D, which would also bisect the angle A, it would appear that the angle BDC is equal to A and half the sum of the angles at the base, or to A and half its supplement. But A, and therefore its supplement, are given in magnitude; and hence BDC is also given in magnitude; therefore (schol. 1, page 272, No. 2) the locus of D is a given circle.

Prop. XIII. Theor.—If from two given points, two straight lines be drawn meeting one another, such that the square of one of them is to the difference between the square of the other and a given space, in a given ratio of inequality, the locus of their point of intersection is the circumference of a given circle.

Let A, B be the given points, and let C be another point such that AC, BC being drawn, the difference between the square of AC and a given space, has to the square of BC a given ratio of inequality: the locus of C is a given circle.

In AB produced, take D (VI. 10, schol.) so that AD has to BD the given ratio; and in AB find E so that the rectangle AB.AE may be equal to the given space; take DF a mean proportional between ED, BD: a circle described from D as centre, with DF as radius, is the locus of C.

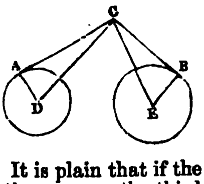


Join CE, and through B, C, E describe a circle cutting AC in G; draw BG, and through ABG describe a circle meeting CB produced in H, and join AH, CD. Then because FD or CD is a mean proportional between ED, BD, ED : DC :: DC : DB. Hence (VI. 6) since the triangles EDC, BDC have a common angle, the angles DEC, DCB are equal.

But (III. 21) the angle DEC or BEC is equal to BGC, and this again (III. 22, cor.) to H: therefore the alternate angles H and DCB are equal, and AH, CD are parallel; wherefore (VI. 2) AD : DB :: HC : CB, or (VI. 1) AD : DB :: HC.CB, or AC.CG : BC^2. But AC.AG is equal to AB.AE, and therefore to the given space; and AC.CG is the difference between the square of AC and the rectangle AC.AG: the difference, therefore, between the square of AC and the given space, has to the square of BC the ratio of AD to DB, which is the given ratio.

Schol. 1. The circle FC is (Schol. page 265, No. 3) the locus of straight lines drawn from E, B, in the ratio of EF to FB, or, as is easily shown, of ED to BD. It is plain also, that if the given ratio be that of equality, the difference of the squares of AC, BC is equal to the given space; and therefore (Schol. page 266, No. 4) the locus of C is a straight line.

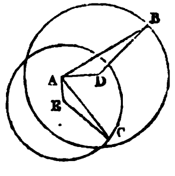
Schol. 2. Hence we have the means of determining the locus of the intersection of tangents AC, BC to two given circles, having to one another a given ratio, that of m to n. For AD, BE being radii drawn to the points of contact, we have the square of AC to the square of BC, that is, $CD^2 - AD^2 : CE^2 - EB^2 :: m^2 : n^2$. Take a space S such that $S : EB^2 :: m^2 : n^2$. Then $CD^2 - AD^2 + S : CE^2 :: m^2 : n^2$; that is, the difference between the square of CD and the given space $AD^2 - S$, has to the square of CE a given ratio; and, therefore, if m, n be unequal, the locus of C is a given circle; if they be equal, it is a straight line. It is plain that if the circles be diminished to points, this becomes the same as the third locus mentioned in the scholium, page 265.



Prop. XIV. Theor.—If two straight lines in a given ratio, and containing a given angle, be drawn from a given point, and if one of them terminate in a given circumference, the other will terminate in another given circumference.

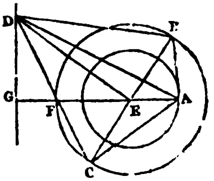
Let A be a given point, and AB a straight line drawn from it to any point B in the circumference of a given circle: if another straight line be drawn through A, making with AB a given angle, and having to it a given ratio, its other extremity is in a given circumference.

Join A and B with D, the centre of the given circle, and make the angle DAE equal to the given angle; make AD to AE in the given ratio; and make the angles EAC, AEC respectively equal to BAD, ADB: the circle of which E is the centre, and EC the radius, is the required locus. For since AD, which is given, has to AE a given ratio, AE is given; and (VI. 4) AD : AB :: AE : AC; and, alternately, AD : AE :: AB : AC; therefore, AB has to AC the given ratio. But



the angles BAD, CAE being equal, add DAC to each; then BAC is equal to DAE, that is, (const.) to the given angle. In the same manner, it would be shown, that any other straight lines drawn through A, one to each circumference, and inclined at the given angle, would be in the given ratio. Therefore the circle of which E is the centre, and EC the radius, is the locus of C.

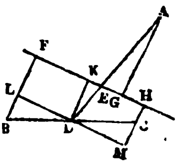
Prop. XV. Theor.—Two concentric circles being given, if from any point A in the circumference of one of them, and from another point D, straight lines be drawn to B, C, the extremities of any diameter of the other circle, so that the squares of AB, AC, DB, DC are together double of the square of DA, the locus of D is a straight line given in position.



Draw AEF through the common centre, and to the radii AE, EF take EG a third proportional; draw GD perpendicular to GA: it is the required locus.

Join ED. Then (II. A) $AB^2+AC^2+DB^2+DC^2=2AE^2+2DE^2+4EC^2=2AE^2+2DE^2+4AE.EG$. But (II. 12) $AD^2=AE^2+DE^2+2AE.EG$; wherefore $AB^2+AC^2+DB^2+DC^2=2AD^2$; and the same may be shown to hold, wherever D is taken in DG. Therefore, DG is the required locus.

Prop. XVI. Porism.—A point may be found such that if, to any straight line whatever passing through it, perpendiculars be drawn from three given points, one of these perpendiculars is equal to the other two.



Let A, B, C be three given points: join BC and bisect it in D; join AD, and take DE a third of it: if any straight line whatever FEGH be drawn through E, the perpendicular AG, drawn from one of the points A to FH, is equal to the two perpendiculars BF, CH, drawn from the points on the other side of FH.

Draw DK perpendicular to FH, and through D draw LDM parallel to FH, and meeting the perpendiculars LB, HC in L, M. Then, in the similar triangles AGE, DKE, AE : AG :: DE : DK; therefore, because AE is double of DE, AG is double of DK. Now, in the equiangular triangles DLB, DMC, BD, DC are equal, and therefore BL, CM are also equal. To each add CH: then LB, CH are equal to HM or KD. Add again FL; then BF, CH are equal to twice DK, that is, to AG.

Schol. It is plain (APP. I. 7, schol.) that E is the point in which the straight lines drawn from the angles of the triangle ABC bisecting the opposite sides would cut one another.

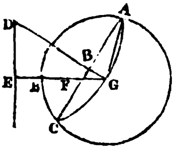
It may be farther remarked, that this porism may be regarded as the case in which the following problem becomes indeterminate: "Through a given point to draw a straight line, such that the perpendicular to it from one of three given points may be equal to the sum of those drawn to it from the other two;"—a problem which is always determinate, except when the point through which the line is to be drawn, is the same as E, determined as above; since in every other case E is to be found in the manner already shown, and the required line is drawn through it and the given point.

PROP. XVII. PORISM.—A circle and a straight line being given, a point may be found within the circle, such that if any chord be drawn through it, the centre of the circle which passes through the extremities of the chord, and through the centre of the given circle, will be in the given straight line.

Let ABC be a given circle, and DE a straight line given in position; a point F may be found within the circle, such that the centre of a circle described through G, the given centre, and through the extremities of any chord AC, passing through F, will be in DE.

Draw GE perpendicular to DE, and take FG such that 2EG : BG :: BG : GF; F is the point through which the chords pass.

Through F let any chord AC, be drawn, and draw GHD perpendicular to it, and join AG. Now (III. 31, and VI. 8, cor.) the square of AG is equal to the rectangle under GH, and twice the radius of a circle passing through A, G, C. But (const.) BG^2 or AG^2 is equal to 2EG.GF; and (VI. 4) DG : GE :: FG : GH; whence (VI. 16, and by doubling) 2DG.GH is equal to 2EG.GF, and (I. ax. 1) 2DG.GH is equal to AG^2. Hence the rectangle under GH and twice DG is equal to the rectangle under GH and twice the radius of the circle passing through A, G, C: therefore DG must be the radius of that circle, and its centre is therefore in DE.

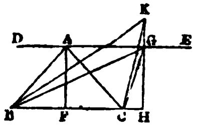


Schol. The property developed in this proposition, might also be expressed in form of the following local theorem: The locus of the centre of a circle described through the centre of a given circle, and the extremities of a chord passing through a given point, is a straight line given in position.

PROP. XVIII. THEOR.—Of isoperimetrical triangles on the same base, the isosceles one is the greatest.

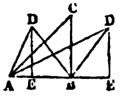
Let ABC be an isosceles triangle on the base BC; it is greater than any other triangle on BC, of equal perimeter, and not isosceles; that is, than any other triangle on BC, having the sum of its other sides equal to the sum of AB, AC.

Draw DAE parallel and AF perpendicular to BC: the altitude of any triangle on BC, having its perimeter equal to that of ABC, and not being isosceles, is less than AF. For, through any other point G, in the parallel draw GB, GC. Then the angles which BA, CA make with DE are equal, being (I. 29) equal to the equal angles ABC, ACB; while those which GB, GC make with it are unequal, being equal to the unequal angles GBC, GCB. Hence (App. I. 12) the sum of BG, GC is greater than the sum of BA, AC. Let GH be perpendicular to BC, and draw BK, CK to any point K in its continuation through G. Then (I. 19) in the triangles BGK, CGK, BK is greater than BG, and CK than CG; wherefore BK, KC are greater than BG, GC, and consequently than BA, AC, which are less than BG, GC. Hence, by adding BC, the triangle BGC, which has its altitude equal to AF; and the triangle BKC, which has its altitude greater than AF, have each a greater perimeter than ABC: and therefore a triangle on BC, which is not isosceles, and which has its perimeter equal to that of ABC, must have its altitude less than AF, and must therefore be less than ABC.*



Prop. XIX. Theor.—If two sides of a triangle be given, the triangle is a maximum, when they contain a right angle.

Let ABC, ABD be triangles on the base AB, and having the sides BC, BD equal: if ABC be a right angle, the triangle ABC is greater than ABD. For (I. 19) the perpendicular DE is less than DB, or than the perpendicular BC; and therefore, since the base AB is common, the triangle ABD is less than ABC.



Prop. XX. Theor.—Of isoperimetrical polygons, having a given number of sides, the maximum is the one which is equilateral.

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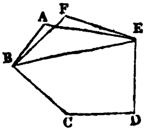
* The demonstration of this proposition given in Legendre's Geometry, Book IV. Appendix, is deficient, as it is assumed without proof, that, in his diagram, the point M is not in the straight line AN.

For, if possible, let ABCDE be the maximum polygon, and yet the side AE greater than AB. On BE describe the isosceles triangle BFE having the sum of its sides BF, FE equal to the sum of BA, AE. Then (App. II. 18) the triangle BFE is greater than BAE, and (I. ax. 4) the polygon FBCDE than ABCDE. Hence the latter is not a maximum, and the same may be shown in a similar manner, unless the sides be all equal.

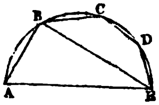


Schol. Hence one requisite in constituting a polygon, of a given perimeter and a given number of sides, a maximum, is that it must be equilateral.

Prop. XXI. Theor.—If all the sides of a quadrilateral or polygon except one be given in magnitude, the figure will be a maximum, when that remaining side is the diameter of a semicircle, and the figure is inscribed in that semicircle.

Let ABCDE be the greatest polygon that can be contained by sides AB, BC, CD, DE given in magnitude, and AE not given: AE is the diameter of a semicircle, and the angular points of the polygon are on the arc of that semicircle.

Join BE. Then, if ABCDE be not a semicircle, and therefore ABE not a right angle, by making it a right angle, the triangle ABE (App. II. 19) would be enlarged, as would also the whole polygon, the magnitude of the part BCDE not being changed by a change in the angle ABE. If the polygon be a maximum, therefore, ABE must be a right angle, and the segment ABE a semicircle. In the same manner it would be shown, by drawing lines from C and D to A and E, that ACE, ADE are right angles; and the proof will be the same if there be six or any greater number of sides; whence the truth of the proposition is manifest.



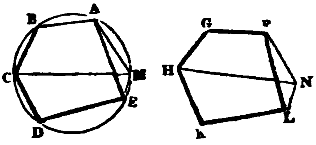
Schol. There is but one semicircle which will contain the maximum polygon. For, suppose ABCDE to be a semicircle containing it; the angles at its centre subtended by the chords AB, BC, CD, DE, amount to two right angles, which would no longer be the case, were the radius made either greater or less; as the triangles having their vertices at the centre, and contained by the radii and the sides AB, BC, &c. would evidently have the magnitudes of their angles changed.

Schol. It is plain that the area of the polygon will be the same in whatever order the sides are arranged; as, however they are placed, they will cut off equal segments, and the polygon is the part of the semicircle remaining, when those segments are taken away.

Prop. XXII. Theor.—Of quadrilaterals and polygons which have their sides equal, each to each, the greatest is that which can be inscribed in a circle.

In the rectilineal figures ABCDE, FGHKL, let the sides AB, BC, CD, DE, EA be severally equal to FG, GH, HK, KL, LF, and let AD be inscribed in a circle, but FK not be capable of being inscribed in one: AD is greater than FK.

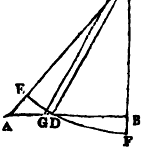
Draw the diameter CM, and join AM, EM; construct the triangle FNL, having the sides FN, NL respectively equal to AM, ME, so that (I. 8) the triangles themselves are equal, and join HN. Then, by the preceding proposition, the figures CBAM, CDEM are respectively greater than HGFN, HKLN; and therefore (I. ax. 4) the polygon ABCDEM is greater than FGHKLN; and, by taking away the equal triangles AME, FNL, there remains the figure ABCDE greater than FGHKL.



Cor. From this proposition and the twentieth it follows, that of polygons of the same number of sides, and of equal perimeters, the regular polygon is the greatest; and that of quadrilaterals of equal perimeters, the greatest is the square.

Prop. XXIII. Theor.—Of regular polygons which have equal perimeters, that which has the greater number of sides is the greater.

Let AB be half the side of the polygon which has the less number of sides, and BC a perpendicular to it, which will evidently pass through the centre of its inscribed or circumscribed circle let C be that centre, and join AC. Then, ACB will be the angle at the centre subtended by the half side AB. Make BCD equal to the angle subtended at the centre of the other polygon by half its side, and from C as centre, with CD as radius, describe an arc cutting AC in E, and CB produced in F. Then, it is plain, that the angle ACB is to four right angles, as AB to the common perimeter; and four right angles are to DCB, as the common perimeter to the half of a side of the other polygon, which, for brevity, call S: then, *ex æquo*, the angle ACB is to DCB, as AB to S. But (VI. 33) the angle ACB is to DCB, as the sector ECF to the sector DCF; and consequently (V. 11) the sector ECF is to DCF, as AB to S, and, by division, the sector ECD is to DCF, as $AB - S$ to S. Now the triangle ACD is greater than the sector CED, and DCB is less than DCF. But (VI. 1) these triangles are as their bases AD, DB; therefore AD has to DB a greater ratio, than $AB - S$ to S. Hence AB, the



sum of the first and second, has to DB, the second, a greater ratio than AB, the sum of the third and fourth, has to S the fourth;* and therefore (V. 10) S is greater than DB. Let then BG be equal to S, and draw GH parallel to DC, meeting FC produced in H. Then, since the angles GHB, DCB are equal, BH is the perpendicular drawn from the centre of the polygon having the greater number of sides to one of the sides; and since this is greater than BC, the like perpendicular in the other polygon, while the perimeters are equal, it follows that the area of that which has the greater number of sides is greater than that of the other.

PROP. XXIV. THEOR. — A circle is greater than any regular polygon of the same perimeter.

For, if a polygon P similar to the one Q proposed for comparison, be described about the circle C: since the area of the circumscribed polygon is evidently equal to the rectangle under its perimeter and half the radius, and (APP. I. 39) the area of the circle equal to the rectangle under its perimeter and half its radius; therefore (VI. 1) P is to C, as the perimeter of P to the perimeter of C, or of Q: and (VI. 20) P is to Q in the duplicate ratio of a side of P to a side of Q, or (VI. 20, cor. 4) in the duplicate ratio of the perimeter of P to the perimeter of Q. Hence P is to a mean proportional between P and Q, as the perimeter of P to the perimeter of Q; and therefore (V. 11) P is to C, as P to a mean proportional between P and Q; that is, (V. 9) C is a mean proportional between P and Q; and (V. 14) being less than P, it is greater than Q, the polygon of equal perimeter.

Schol. Every thing here proved will evidently hold equally regarding a circle C, as compared with *any* polygon Q of equal perimeter, which is such that a circle can be inscribed in it; for then a polygon P, similar to Q, can be described about C.

Cor. Hence it appears also, that a circle is a mean proportional between *any* rectilineal figure described about it, and a similar one of equal perimeter with the circle.

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* This may be proved by dividing the first term by the second, and the third by the fourth, and adding unity to each quotient. See the Supplement to Book V.

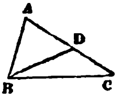
BOOK III.

PROPOSITIONS.

Prop. I. Prob.—Given the base of a triangle, the difference of the sides, and the difference of the angles at the base; to construct it.

Make BC equal to the given base, and CBD equal to half the difference of the angles at the base: from C as centre, at a distance equal to the difference of the sides, describe an arc cutting BD in D: join CD and produce it: make the angle DBA equal to BDA; ABC is the required triangle.

For (I. 6) AD is equal to AB, and the difference of AC, AD, or of AC, AB, is CD; and (App. I. 6) since AD is equal to AB, CBD is equal to half the difference of the angles at the base. The triangle ABC, therefore, is the one required, as it has its base equal to the given base, the difference of its sides equal to the given difference, and the difference of the angles at the base equal to the given difference of those angles.

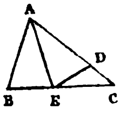


Method of Computation. In the triangle BCD, there are given BC, CD, and the angle CBD; whence the angle C can be computed; and the sum of this and twice CBD is the angle ABC. Then, in the whole triangle ABC, the angles and the side BC are given; whence the other sides may be computed: or, one of them being computed, the other will be found by means of the given difference CD.

Prop. II. Prob.—Given the segments into which the base of a triangle is divided by the line bisecting the vertical angle, and the difference of the sides; to construct the triangle.

Construct the triangle CED, having the sides CE, ED equal to the given segments, and CD equal to the given difference of the sides: produce CE, and make EB equal to ED; bisect the angle BED by EA, meeting CD produced in A, and join AB: ABC is the required triangle.

For, in the triangles AEB, AED, BE is equal to ED, EA common, and the angle BEA equal to DEA: therefore (I. 4) BA is equal to DA, and the angle EAB to EAD. Hence, ABC is the required triangle; for CD, the difference of its sides, is equal to the given difference, and BE, EC, the segments into

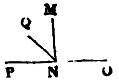


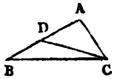
which the base is divided by the line bisecting the vertical angle, are equal to the given segments.

Method of Computation. The sides of the triangle CDE are given, and therefore its angles may be computed: one of which, and the supplement of the other are the angles C and B. Then, in the triangle ABC, the angles and BC are given, to compute the remaining sides.

OTHERWISE: — Since (VI. 3) CE : EB :: CA : AB, we have, by division, CE — EB : EB :: CA — AB : AB; which, therefore, becomes known, since the first three terms of the analogy are given; and thence AC will be found by adding to AB the given difference of the sides.

PROP. III. PROB. — Given the base of a triangle, the vertical angle, and the difference of the sides; to construct the triangle.

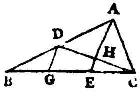




Let MNO be the given vertical angle; produce ON to P, and bisect the angle MNP by NQ. Then, make BD equal to the difference of the sides, and the angle ADC equal to QNP; from B as centre, with the given base as radius, describe an arc cutting DC in C; and make the angle DCA equal to ADC: ABC is the required triangle. For (const.) the angles ACD, ADC are equal to MNP, and therefore (I. 32, and 13) the angle A is equal to MNO. But (I. 6) AD is equal to AC, and therefore BD is the difference of the sides AB, AC; and the base BC is equal to the given base: wherefore ABC is the triangle required.

Method of Computation. In the triangle CBD, the sides BC, BD, and the angle BDC, the supplement of ADC or MNQ are given; whence the other angles can be computed. The rest of the operation will proceed as in the first proposition of this book.

PROP. IV. PROB.—Given one of the angles at the base of a triangle, the difference of the sides, and the difference of the segments into which the base is divided by the line bisecting the vertical angle; to construct the triangle.

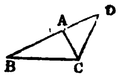


Construct the triangle DBG, having the angle B equal to the given angle, BD equal to the difference of the sides, and BG equal to the difference of the segments; draw DC perpendicular to DG, and meeting BG produced in C; produce BD, and make the angle DCA equal to CDA: ABC is the triangle required. For it has B equal to the given angle, and the difference of its sides BD equal to the given difference: and if AHE be drawn bisecting the angle BAC, it bisects (I. 4) CD, and is perpendicular to it: it is therefore parallel to DG, one side of the triangle CDG; and, bisecting CD in H, it also (VI. 2) bisects GC in E. Hence BG, the difference of BE, GE, is also the difference of BE, EC, the segments into which the base is divided by the line bisecting the vertical angle.

Method of Computation. In the triangle DBG, BD, BG, and the angle B are given; whence (Trig. 3) we find half the difference of the angles BGD, BDG, which is equal to half the angle C.* Then (by the same proposition) we have in the triangle ABC, $\tan \frac{1}{2}(C-B) : \tan \frac{1}{2}(C+B) :: c-b$ or $BD : c+b$; whence the sides c and b become known, and thence BC by the first case.

PROP. V. PROB. — Given the base of a triangle, the vertical angle, and the sum of the sides; to construct it.

Make BD equal to the sum of the sides, and the angle D equal to half the vertical angle; from B as centre, with the base as radius, describe an arc meeting DC in C;† and make the angle DCA equal to D; ABC is the required triangle.



For (I. 6) AD is equal to AC, and therefore BA, AC are equal to BD, the given sum. Also (I. 32) the exterior angle BAC is equal to the two D and ACD, or to the double of D, because D and ACD are equal: therefore, since D is half the given vertical angle, BAC is equal to that angle. The triangle ABC, therefore, has its base equal to the given base, its vertical angle equal to the given one, and the sum of its sides equal to the given sum: it is therefore the triangle required.

Method of Computation. In the triangle BDC, the angle D, and the sides BC, BD are given: whence the remaining angles can be computed: and then, in the triangle ABC, the angles and the side BC are given, to compute the other sides.

PROP. VI. PROB.—Given the vertical angle of a triangle, and the segments into which the line bisecting it divides the base; to construct it.

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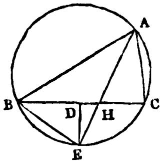
* For (I. 32) $BEA = \frac{1}{2}A + C$. and consequently $BEA - \frac{1}{2}A = C$. But (I. 29) $BGD = BEA$, and $BDG = BAE = \frac{1}{2}A$; and therefore $BGD - BDG = C$.

† Should the circle neither cut nor touch DC, the problem would be impossible with the proposed data. If the circle meet DC in two points, there will be two triangles each of which will answer the conditions of the problem. These triangles, however, will differ only in position, as they will be on equal bases, and will have their remaining sides equal, each to each. This problem might also be solved by describing (III. 33) on the given base BC a segment of a circle containing an angle equal to half the vertical angle; by inscribing a chord BD equal to the sum of the sides; by joining DC; and then proceeding as before. The construction given above is preferable.

In the straight line BC, take BH and CH equal to the segments of the base; on BC describe (III. 33) the segment BAC containing an angle equal to the vertical angle, and complete the circle; bisect the arc BEC in E; draw EHA, and join BA, CA: ABC is the required triangle. For (III. 27) the angles BAH, CAH are equal, because the arcs BE, EC are equal: and therefore the triangle ABC manifestly answers the conditions of the question.[*]

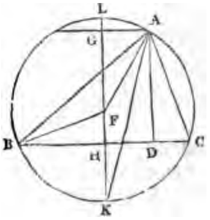


Method of Computation. Join BE, and draw ED perpendicular to BC. Then BD or DC is half the sum of the segments BH, HC, and DH half their difference: and BD is to DH, or twice BD to twice DH, as tan DEB to tan DEH. Now it is easy to show that BED is half the sum of the angles ABC, ACB, and HED half their difference; and therefore these angles become known; and BC being given, the triangle ABC is then resolved by the method for the first case.

Cor. Hence, we have the method of solving the problem in which *the base, the vertical angle, and the ratio of the sides of a triangle are given, to construct it.* For (VI. 3) the sides being proportional to the segments BH, HC, it is only necessary to divide the given base into segments proportional to the sides, and then to proceed as above.

PROP. VII. PROB.—Given the base, the perpendicular, and the vertical angle of a triangle; to construct it.

Make BC equal to the given base, and (III. 33) on it describe a segment capable of containing an angle equal to the vertical angle; draw AG parallel to BC, at a distance from it equal to the given perpendicular, and meeting the arc in A; join AB, AC; ABC is evidently the triangle required.



Method of Computation.—Draw the perpendicular AD, and parallel to it draw LGHK through the centre F; join BF, AF, AK. Now, since AK evidently bisects the angle BAC, the angle KAD or K is (APP. I. 6) equal to half the difference of the angles ABC

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[*] The construction might also be effected by describing on BH and CH segments each containing an angle equal to half the vertical angle, and joining their point of intersection A with B and C. Another solution may be obtained by the principle (VI. 3) that BA : AC :: BH : HC. For if a triangle be constructed having its vertical angle equal to the given one, and the sides containing it equal to the given segments, or having the same ratio, that triangle will be similar to the required one; and therefore on CB construct a triangle equiangular to the one so obtained.

A fourth solution may be obtained from proposition G of the sixth book; but it is not so easy as those already pointed out.

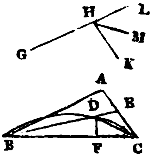
ACB, and therefore (III. 20) AFG is the whole difference of these angles. Then, in the right-angled triangle BHF, the angles and BH being known, FH can be computed; from which and from AD or GH, FG becomes known. Now, to the radius BF or AF, FH is the cosine of BFH or BAC, and FG the cosine of AFG; and therefore FH : FG : : cosBAC : cosAFG. Hence the angles ABC, ACB become known, and thence the remaining sides.

Schol. Should the parallel AG not meet the circle, the solution would be impossible, as no triangle could be constructed having its base, perpendicular, and vertical angle of the given magnitudes. If the parallel cut the circle, there will be two triangles, either of which will answer the conditions of the question. They will differ, however, only in position, as their sides will be equal, each to each. If the parallel touch the circle, there will be only one triangle; and it will be isosceles.

PROP. VIII. PROB.—Given the base of a triangle, the vertical angle, and the radius of the inscribed circle; to construct the triangle.

Let GHK be the given angle; produce GH to L, and bisect LHK by HM: on the given base BC describe (III. 33) the segment BDC, containing an angle equal to GHM: draw a straight line parallel to BC, at a distance equal to the given radius, and meeting the arc of the segment in D: join DB, DC; and make the angles DBA, DCA equal to DBC, DCB, each to each: ABC is the required triangle.

Produce BD to E, and draw DF perpendicular to BC. Then, since (const.) the angle BDC is equal to GHM, the two DBC, DCB are equal (I. 32 and 13) to LHM, and therefore (const.) ABD, ACD are equal to KHM. But (I. 32) BDC is equal to BEC, ECD, or to BAC, ABD, ACD, because (I. 32) BEC is equal to BAC, ABD. Therefore BAC, ABD, ACD are equal to GHM: from the former take ABD, ACD, and from the latter KHM, which is equal to them, and the remainders BAC, GHK are equal. It is plain also (I. 26) that perpendiculars drawn from D to AB and AC, would be each equal to DF; and therefore a circle described from D as centre, with DF as radius, would be inscribed in the triangle ABC; and BC being the given base, and A being equal to the given vertical angle, ABC is the required triangle.

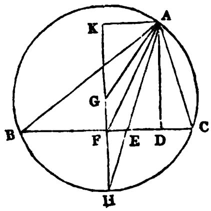


The method of computation is easily derived from that of the preceding proposition.

Schol. Should the parallel to BC not meet the arc of the segment, the solution would be impossible, as there would be no triangle which could have its base, its vertical angle, and its inscribed circle of the given magnitudes. If the parallel be a tangent to the arc, the radius of the inscribed circle would be a maximum. Hence, to solve the problem in which the base and the vertical angle are given, to construct the triangle, so that the inscribed circle may be a maximum, describe the segment as before, and to find D bisect the arc of the segment. The rest of the construction is the same as before; and the triangle will evidently be isosceles.

PROP. IX. PROB.—Given the three lines drawn from the vertex of a triangle, one of them perpendicular to the base, one bisecting the base, and one bisecting the vertical angle; to construct the triangle.[*]

Take any straight line BC and draw DA perpendicular to it, and equal to the given perpendicular; from A as centre, with radii equal to the lines bisecting the vertical angle and the base, describe arcs cutting BC in E and F, and draw AEH and AF; through F draw GFH perpendicular to BC, and draw AG making the angle HAG equal to H, and cutting HG in G: from G as centre, with GA as radius, describe a circle cutting BC in B and C: join AB, AC: ABC is the triangle required.



For (III. 3) since GFH is perpendicular to BC, BC is bisected in F; and (III. 30) the arcs BH, HC are equal. Therefore (III. 27) the angles BAH, CAH are equal. Hence, in the triangle ABC, the perpendicular AD, the line AE bisecting the vertical angle, and the line AF bisecting the base, are equal to the given lines. Therefore ABC is the triangle required.

Schol. If the three given lines be equal, the problem is indeterminate; as any isosceles triangle, having its altitude equal to one of the given lines, will answer the conditions.

Method of Computation. Through A draw a parallel to BC, meeting HG produced in K. Then, in the right-angled triangle ADE, AE, AD being given, DAE, or H, may be computed; the double of which is AGK: and AK or FD being given, AG, GK can be found, and thence GF. Hence, if GB were drawn, it and GF being known, BF, and the angle BGF, or BAC, can be computed. The rest is easy; DAE, half the difference of B and C, being known.

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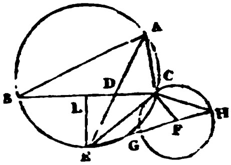
[*] *Analysis.* Let ABC be the required triangle, AD the perpendicular, and AE, AF the lines bisecting the vertical angle and the base. About ABC describe (IV. 5) a circle, and join its centre G, with A and F, and produce GF to meet the circumference in H. Then (III. 3) GFH is perpendicular to BC, and (III. 30) the arcs BH, HC are equal. But (III. 26) the equal angles BAE, CAE at the circumference stand on equal arcs; and therefore AE being produced, will also pass through H: and the point H, and the angle GHA and its equal HAG are given. Hence also the centre G and the circle are given, and the method of solution is plain.

PROP. X. PROB.—Given the base of a triangle, the vertical angle, and the straight line bisecting that angle; to construct the triangle.[*]

On the given base BC describe (III. 33) the segment BAC capable of containing an angle equal to the given vertical angle, and complete the circle; bisect the arc BEC in E, and join EC; perpendicular to this, draw CF equal to half the line bisecting the vertical angle, and from F as centre, with FC as radius, describe the circle CGH, cutting the straight line passing through E and F in G and H; make ED equal to EG, and draw EDA; lastly, join AB, AC, and ABC is the required triangle.



For the triangles CEA, CED are equiangular, the angle CEA being common, and BCE, CAE being each equal to BAE. Therefore AE : EC :: EC : ED, and (VI. 17) AE.ED$=$EC^2. But (III. 18 and 36) HE.EG or HE.ED$=$EC^2; and therefore AE.ED$=$HE.ED; whence AE$=$HE, and (I. ax. 3) AD$=$GH$=$2CF. AD is therefore equal to the given bisecting line, and it bisects the angle BAC. Hence ABC is the required triangle.

Method of Computation. Draw EL perpendicular to BC, and join CH. Then BCE is equal to BAE, half the vertical angle A; and therefore, to the radius EC; CL is the cosine of ½A, and CF is the tangent of CEF to the same radius: wherefore, to any radius, CL : CF, or BC : AD :: cos½A : tan CEF or cot EFC; and hence the angle H, being half of EFC, is known. Also ECH is the complement of H, because ECF is a right angle, and FCH equal to H. But (APP. III. 2) EC : EH or EA :: sin H : sin ECH, or cos H; or (APP. III. defs. cor. 6) EC : EA :: R : cot H. Also in the triangle ACE, EC : EA :: sin½A : sin ACE; whence (V. 11) R : cot H :: sin½A : sin ACE; whence ACE may be found; and if from it, and from ABE, its supplement (BE being supposed to be joined) BCE be taken, the remainders are the angles at the base.

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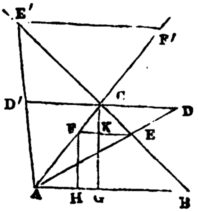
[*] *Analysis.* Let ABC be the required triangle, and let AD, the line bisecting the vertical angle, be produced to meet the circumference of the circumscribed circle in E; join also EC. Then (III. 33) the circumscribed circle is given, since the base and vertical angle are given; and the arc BEC is given, as are also its half EC, and the chord EC. Now the triangles AEC, DEC are equiangular; for the angle CEA is common, and (III. 21) BCE is equal to BAE or EAC. Hence AE : EC :: EC : ED, and therefore AE.ED$=$EC^2. Hence (III. 36) it is evident that if EC be made a tangent to a circle, and if through the extremity of the tangent a line be drawn cutting the circle, so that the part within the circle may be equal to AD, DE will be equal to the external part: whence the construction is manifest.

Prop. XI. Prob.—**To draw a line parallel to a given straight line, such that the part of the parallel intercepted between two other straight lines given in position, may have a given ratio to the perpendicular distance between the two parallels.**

Let AB be a given straight line falling on two other given straight lines AC, BC; it is required to draw a parallel to AB, such that the part of it between AC, BC may have a given ratio to the perpendicular distance between AB and the parallel.



Through C draw CG perpendicular and CD parallel to AB, and take CD to CG in the given ratio; draw AED, and through E draw EF parallel to AB: EF is the line required.

Draw FH parallel to CG. Then, in the similar triangles, ACD, AFE, and ACG, AFH, CD : CA :: FE : FA, and CA : CG :: FA : FH; whence, ex æquo, CD : CG :: FE : FH. FE, therefore, has to FH the given ratio, and it is parallel to AB.

Schol. 1. If DC be produced, and CD′ be taken equal to CD, and if AD′E′ be drawn, the straight line E′F′ parallel to AB, would be proved in the same manner, to have the given ratio to its perpendicular distance from AB: it therefore equally answers the conditions of the problem. If, however, CD′ be equal to AB, the second solution is inadmissible, as (I. 33) AD′, BC would not meet. If CD′ be greater than AB, E′F′ will be on the other side of AB.

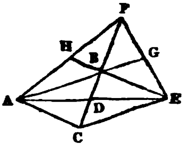
Schol. 2. If CD, CG, and consequently FE, FH, be equal, and if a parallel to FH were drawn through E, the figure EH would be a square; and we have thus the means of solving the problem in which it is required *to inscribe a square in a given triangle, having one of its sides on an assigned side of the triangle.* Now, since AB is to FE, and CG to CK, as CA to CF, we have AB : FE :: CG : CK; whence, alternately, and by inversion, CG : AB :: CK : FE, and, by composition, AB + CG : AB :: CK + FE : FE; that is, in case of the inscribed square, AB + CG : AB :: CG : FE; and, therefore, the side of the inscribed square is a fourth proportional to the sum of the base and perpendicular, and to the base and perpendicular themselves.

It would be proved in a similar manner, that when CG, CD′ are equal, E′F′ would be a fourth proportional to the *difference* of the base and perpendicular, and to the base and perpendicular themselves.

Cor. BE′ is divided harmonically (App. I. 29) in C and E. For DD′, a side of the triangle ADD′, is bisected in C, and AB is drawn through A parallel to DD′. It would be shown in a similar manner, that AF′ is divided harmonically in F and C.

Prop. XII. Prob.—Given the straight lines drawn from the three angles of a triangle to the points of bisection of the opposite sides; to construct the triangle.

Trisect the three given lines, and describe the triangle ABC having its three sides respectively equal to two thirds of the three given lines; complete the parallelogram ABEC, and draw the diagonal AE; produce also CB, making BF equal to BC; and join FA, FE: AFE is the required triangle.



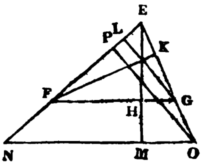
Produce AB, EB to G, H. Then, (II. B, cor.) AE, BC bisect each other in D, and, therefore, FD is equal to one of the given lines, for BD is one third of it, and FB two thirds. Again, because FB, BC are equal, and HB parallel to AC, FA is bisected in H, and HB is half of AC or BE. Hence, HE is equal to another of the given lines, and it bisects FA. In the same manner it would be proved, that AG is equal to the remaining line, and that it bisects FE. Hence FAE is the triangle required.

Method of Computation. BD, which is a third of one of the given lines, bisects AE, a side of the triangle ABE, in which the sides AB, BE are respectively two thirds of the two remaining lines. Then (II. A) $2AD^2 = AB^2 + BE^2 - 2BD^2$; whence AD, and consequently AE may be found: and in the same manner the other sides may be computed.

Prop. XIII. Prob.—Given the three perpendiculars of a triangle; to construct it.

Let A, B, C be three given straight lines; it is required to describe a triangle having its three perpendiculars respectively equal to A, B, C.

Take any straight line D, and describe a triangle EFG, having the sides FG, FE, EG third proportionals to A and D, B and D, C and D; and draw the perpendiculars EH, GL, FK.



Then the rectangles FG.A, EF.B are equal, each being equal to the square of D; and therefore EF : FG :: A : B. But in the similar triangles EHF, GLF, EF : FG :: EH : GL; wherefore EH : GL :: A : B; and in the same manner it would be proved, that EH : FK :: A : C. Hence (V. 14) if EH be equal to A, GL is equal to B, and FK to C; and EFG is the triangle required.

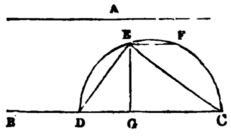
But if EH be not equal to A, make EM equal to it, and draw NMO parallel to FG, and meeting EF, EG, produced, if necessary, in N and O: ENO is the required triangle. Draw OP perpendicular to EN. Then EM : EH :: EO : EG, and OP : GL :: EO : EG; whence (V. 11, and alternately) EM : OP :: EH : GL; or, by the foregoing part, EM : OP :: A : B; wherefore (V. 14) since EM is equal to A, OP is equal to B; and it would be proved in a similar manner, that the perpendicular from N to EO is equal to C.

Method of Computation. By dividing any assumed number successively by A, B, C, we find the sides of the triangle EFG, and thence its angles, or those of ENO; whence, since the perpendicular EM is given, the sides are easily found.*

PROP. XIV. PROB.—Given the sum of the legs of a right-angled triangle, and the sum of the hypotenuse, and the perpendicular to it from the right angle; to construct the triangle.

Let the sum of the legs of a right-angled triangle be equal to the straight line A, and the sum of the hypotenuse and perpendicular equal to BC: it is required to construct the triangle.

Find (I. 47, cor. 3) a straight line the square of which is equal to the excess of the square of BC above that of A, and cut off BD equal to that line; on DC as diameter describe a semicircle, and draw EF parallel to DC at a distance equal to BD; join either point of intersection, E, with D and C: DEC is the required triangle.



Draw the perpendicular EG, which (const.) is equal to BD. Then (II. 4) BC^2 or $A^2+EG^2=(DC+EG)^2=DC^2+EG^2+2DC.EG$; whence $A^2=DC^2+2DC.EG$. Also $(DE+EC)^2=DE^2+EC^2+2DE.EC=DC^2+2DC.EG$, because (III. 31, and I. 47) $DC^2=DE^2+EC^2$, and $DC.GE=DE.EC$, each being equal to twice the area of the triangle DEC. Hence $(DE+EC)^2=A^2$; wherefore (I. 46, cor. 3) $DE+EC=A$; and DEC is the triangle required.

Method of Computation. From the construction, we have BD or $EG=\sqrt{(BC^2-A^2)}$. Then $DC=BC-BD$; by halving which we get the radius of the semicircle: and if from the square of the radius drawn from E, the square of EG be taken, and if the square root of the remainder be successively taken from the radius, and added to it, the results will be the segments DG, GC; from which, and from EG, the sides (I. 47, cor. 1) are readily computed.

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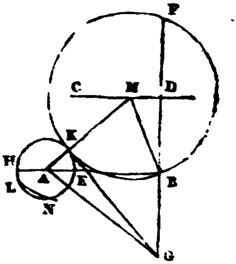
* Or, when the sides of EFG are found, its perpendicular EH may be computed in the manner pointed out in the note to the 13th proposition of the second book. Then EH : A :: FG : NO :: EF : EN :: EG : EO.

Prop. XV. Prob.—Given the base of a triangle, the perpendicular, and the difference of the sides; to construct it.

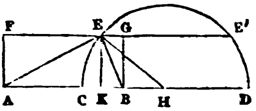
Make AB equal to the given base, and parallel to it draw CD, at a distance equal to the given perpendicular; draw BDF perpendicular to CD, and make DF equal to DB; from A as centre, with a radius AE equal to the given difference, describe the circle ELN; through B, F describe any circle cutting ELN in L, N, and let G be the point in which a straight line drawn through L, N cuts FB produced: draw the tangent GK, and draw AKM cutting CD in M; join BM; and it is evident, from the first corollary to the second proposition of the second book of this Appendix, that AMB is the required triangle.



Method of Computation. From M as centre, with MK as radius describe a circle, and by the corollary referred to, it will pass through B and F. Join AG, and produce BA to H. Then the rectangle $FG.GB = AG^2 - AK^2$, each being equal to the square of GK; that is, $FB.BG + BG^2 = AB^2 + BG^2 - AK^2$. Take away BG^2; then $FB.BG = AB^2 - AK^2 = (AB + AK)(AB - AK) = HB.EB$. Hence BG becomes known. Then, in the two right-angled triangles GBA, GKA, the angles at A can be computed, and their difference is the angle MAB in the required triangle. The rest is easy, if the perpendicular from M to AB be drawn.

Prop. XVI. Prob.—Given the base, the area, and the ratio of the sides of a triangle; to construct it.

Let AB be the given base, and (VI. 10, and schol.) find the points C, D, so that AC, CB and AD, DB are in the ratio of the sides; on CD as diameter describe the semicircle CED, and (I. 45, cor.) to AB apply a parallelogram BF double of the given area; let FG, the side of this opposite to AB, produced if necessary, cut the semicircle in E, and join EA, EB : EAB is the required triangle.



For (VI. G, cor.) AE is to EB, as AC to CB; that is, in the given ratio. Also (I. 41) AEB is half of the parallelogram BF, which is double of the given area. Therefore AEB is on the given base, is of the given area, and has its sides in the given ratio.*

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* See the Notes at the end of the volume.

Schol. If FG, or FG produced, do not meet the semicircle, the problem is impossible: if it cut it in E and E′, there will be two triangles essentially different, each of which will answer the conditions of the problem: if it touch the semicircle there will be only one triangle, and it will be the greatest possible with the base of the given magnitude, and the sides in the given ratio; and hence we have the means of solving the problem in which it is required *to construct a triangle on a given base, having its sides in a given ratio, and its area a maximum.*

Method of Computation. Join the centre H with E, and draw the perpendicular EK. Then, let $m : n :: \text{AC} : \text{CB}$, and consequently $m : n :: \text{AD} : \text{DB}$; then, from these, by composition and division, we get $m + n : n :: \text{AB} : \text{CB}$, and $m - n : n :: \text{AB} : \text{DB}$; whence DC and its half, the radius of the circle, become known. EK also is found by dividing double of the area by AB. Then, in the triangle EKH, KH can be found, and thence AK and KB; and, by means of them and EK, the sides EA, EB may be computed. If E′ be taken as the vertex, the method of computation is almost the same, and is equally easy.

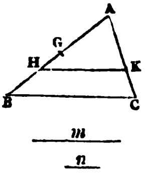
PROP. XVII. PROB. — Given the base of a triangle, the vertical angle, and the rectangle of the sides; to construct it.

On the given base describe a segment containing an angle equal to the given angle; to the diameter of the circle of which this segment is a part, and to the lines containing the given rectangle, find a fourth proportional: this proportional (VI. D) is the perpendicular of the triangle; and the rest of the solution is effected by means of the seventh proposition of this book.

PROP. XVIII. PROB. — To divide a given triangle into two parts in a given ratio, by a straight line parallel to one of the sides.

Let ABC be a given triangle; it is required to divide it into two parts in the ratio of the two straight lines, m, n, by a straight line parallel to the side BC.

Divide (VI. 10) AB in G, so that $\text{BG} : \text{GA} :: m : n$, and between AB, AG find (VI. 13) the mean proportional AH; draw HK parallel to BC; ABC is divided by HK in the manner required.



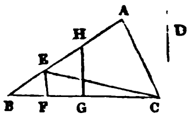
For (VI. 19, cor.) since the three straight lines AB, AH, AG are proportionals: AB is to AG, as the triangle ABC to AHK; whence by division, BG is to AG, or (const.) m is to n, as the quadrilateral BCKH to the triangle AHK.

In practice, the construction is easily and elegantly effected, when the triangle is to be divided either into two or more parts proportional to given lines, by dividing AB into parts proportional to those lines, and through the points of section drawing perpendiculars to AB, cutting the arc of a semicircle described on AB as diameter: then by taking lines on AB, terminated at A, and severally equal to the chords drawn from A to the points of section of the arc, the points on AB will be obtained through which the parallels to BC are to be drawn. The reason is evident from the foregoing proof, in connexion with the corollary to the eighth proposition of the sixth book.

Prop. XIX. Prob.—To divide a given triangle into two parts in a given ratio, by a straight line parallel to a given straight line.

Let ABC be a given triangle, and D a given straight line; it is required to divide ABC into parts in a given ratio, by a straight line parallel to D.

Divide AB in E, so that AE, EB may be in the given ratio, and draw EF parallel to D; take BG a mean proportional between BF and BC, and draw GH parallel to EF: GH divides ABC in the manner required.

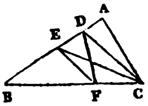


Join EC. Then (VI. 19, cor.) BF : BC :: BFE : BGH, and (VI. 1) BF : BC :: BFE : BCE; wherefore BFE : BGH :: BFE : BCE; and therefore BGH is equal to BEC; and consequently the remainders AHGC, AEC are equal. But (VI. 1) BE : EA :: BEC : AEC; therefore also BE : EA :: BGH : AHGC.

Should the point H not be in BA, but in its continuation, let CA, and not BA, be divided in the given ratio, and the rest of the construction will continue the same, the point C being used instead of B.

Prop. XX. Prob.—To divide a triangle into two parts in a given ratio, by a straight line drawn through a given point in one of the sides.

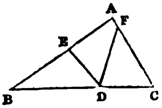
Let ABC be the given triangle, and D the given point; and let AE be to EB in the given ratio; join DC, draw EF parallel to it, and join DF: ABC is divided by DF in the given ratio.



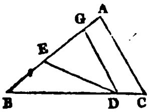
Join EC. Then (I. 37) the triangles EFD, EFC are equal; to each add BEF; then BDF, BEC are equal; and taking these separately from ABC, the remainders ADFC, AEC are equal. Now (VI. 1) CEA, CEB are in the ratio of AE to EB, that is, in the given ratio; and therefore ADFC, BDF are also in the given ratio.

Prop. XXI. Prob.—From a given point in one of the sides of a given triangle, to draw two straight lines trisecting the triangle.

Let ABC be the given triangle, and D the given point; then, if BD be not less than one third of BC, to BD, to a third part of BC, and to the perpendicular from A to BC, find a fourth proportional, and at a distance equal to it draw a parallel to BC, cutting BA in E, and join ED: the triangle BED is evidently (VI. 16, and I. 41) a third part of ABC: and CDF will be constructed in a similar manner, if CD be not less than a third of BC.



If DC, one of the segments, be less than a third of BC, the triangle BDE is constructed as before, but the rest of the preceding solution fails, as the second parallel would fall above the triangle. In this case, take EG equal to EB; then, if DG be joined, the triangle ABC is trisected by DE, DG.

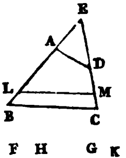


Schol. It is easy to see that this method may be readily extended to the division of a triangle into more equal parts than three, or into parts proportional to given magnitudes, by straight lines drawn from a given point in one of the sides.

PROP. XXII. PROB.—To divide a given quadrilateral into two parts in a given ratio, by a straight line parallel to one of its sides.

Let it be required to divide the quadrilateral ABCD into two parts in a given ratio, by a straight line parallel to the side BC.

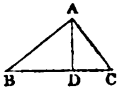
Produce BA, CD to meet in E, and (I. 45) describe a rectangle equal to ABCD. Let FG, a side of that rectangle, be divided in H, so that FH is to HG as the required part next BC to the other part; and to one of the adjoining sides of the same rectangle, apply (I. 44) a rectangle equal to the triangle ADE. Let GK be a side of the latter rectangle; and (APP. IV. 18) draw LM parallel to BC, dividing the triangle EBC, so that ELM : LBCM :: KH : HF; LM obviously divides the quadrilateral in the given ratio.



If BA, CD be parallel, the foregoing solution fails. In that case, the solution is effected simply by describing (I. 44.) on BC a parallelogram equal to the one on FH, and having two of its sides coinciding with BA, CD.

PROP. XXIII. THEOR.—If the sides of a right-angled triangle be continual proportionals, the hypotenuse is divided in extreme and mean ratio by the perpendicular to it from the right angle; and the greater segment is equal to the less or remote side of the triangle.

Let ABC be a triangle right-angled at A, and let AD be perpendicular to BC; then if CB : BA :: BA : AC, BC is divided in extreme and mean ratio in D, and BD is equal to AC.



For (hyp.) CB : BA :: BA : AC, and (VI. 8, cor.) CB : BA :: BA : BD; therefore BA : AC :: BA : BD, and AC is equal to BD. Again, (VI. 8, cor.) BC : CA :: CA : CD, or BC : BD :: BD : DC, and therefore (VI. def. 3) BC is divided in extreme and mean ratio in D.

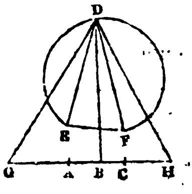
Schol. Conversely, if BC : BD :: BD : CD, and if BAC be a right angle, and DA perpendicular to BC; CB : BA :: BA : AC, and BD is equal to AC. For (hyp.) CB : BD :: BD : CD, and (VI. 8, cor.) CB : CA :: CA : CD; wherefore BD^2 is equal to CA^2, each (VI. 17) being equal to the rectangle BC.CD, and therefore BD is equal to CA. Again, (VI. 8) CB : BA :: BA : BD or AC.

Prop. XXIV. Prob.—Given the angles and diagonals of a parallelogram; to construct it.

On one of the diagonals describe a segment of a circle containing an angle equal to the given angle at either extremity of the other; from the middle point of this diagonal as centre, with half the other diagonal as radius, describe an arc cutting the arc of the segment; through the extremities of the first diagonal draw four straight lines, two to the intersection of the arcs, and two parallel to these: the parallelogram thus formed is easily proved to be the one required.

Prop. XXV. Prob.—Given the vertical angle of a triangle, and the radii of the circles inscribed in the parts into which the triangle is divided by the perpendicular; to construct the triangle.

Take any straight line ABC, and through any point B draw the perpendicular BD; make BA, BC equal to the given radii, and let E, F be the angular points, remote from B, of squares described on AB, BC; join EF, and on it describe the segment EDF, containing an angle equal to half the given vertical angle; let the perpendicular cut the arc EDF in D, and join DE, DF; draw DG, DH making the angles EDG, FDH respectively equal to EDB, FDB; DGH is the required triangle.

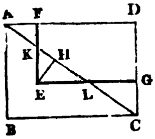


For (I. 26) perpendiculars drawn from E to DB, DG are equal, and each of them is equal (const.) to the perpendicular from E to GB. Each of them therefore is equal to the given radius AB, and a circle described from E at the distance of one of these is inscribed in the triangle DGB. In the same manner it would be shown, that a circle described from F as centre, with the other given radius, would be inscribed in DBH. Hence, since the angle GDH is double of EDF, GDH is equal to the given vertical angle, and the triangle GDH answers the conditions of the question.

Schol. The preceding solution is strictly in accordance with the enunciation, taken in its limited sense. There will be interesting variations, however, if we regard the given circles, not merely as *inscribed*, but as those which touch all the sides of each of the right-angled triangles, either internally or externally. These variations will be obtained, by giving the squares on the radii every possible variety of position in the four right angles formed by the intersection of AC, DB; and the solution will obtain complete generality by taking into consideration both the points in which BD cuts the circle of which EF is a chord.

Prop. XXVI. Prob.—To bisect a given rectangle by two straight lines parallel to two of its adjacent sides, and equally distant from them.

Let ABCD be the rectangle, and AC one of its diagonals; find (IV. 4) E the centre of the circle inscribed in the triangle ABC, and through E draw EKF, ELG parallel to BA, BC: these bisect the rectangle.



Draw EH perpendicular to AC. Then the triangles AFK, EHK are equiangular, and the side AF, being equal to the radius of the inscribed circle, is equal to EH. Hence (I. 26) the triangles AFK, EHK are equal: and it would be proved in a similar manner, that CGL, LHE are equal; and consequently the two AFK, CGL are equal to KEL. From the triangle ABC take KEL, and to the remainder add AFK, CGL; and from ADC take AFK, CGL, and to the remainder add KEL: then (I. ax. 3, and 2) the figure ABCGEF is equal to the rectangle FG; that is, the given rectangle is bisected by EF, EG.

Prop. XXVII. Prob.—Let the four points A, B, C, D, lying in the same straight line, be given in position; to find another point, such that if straight lines be drawn from it to A, B, C, D, the angles at that point subtended by AB, and CD, may be equal to one another, and the one subtended by BC may be equal to a given angle.*

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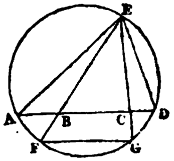
* *Analysis.* Suppose E to be the required point, and join EA, ED; describe the circle AED, and draw EBF, ECG, and FG. Then, because the angles AEF, DEG are equal, the arcs AF, DG are (III. 26) equal, and (III. 26, note to cor.) FG is parallel to AD. Hence (VI. 2) EB : BF :: EC : CG, and therefore (VI. 1, and III. 35) EB^2 : EB.BF or AB.BD :: EC^2 : EC.CG or AC.CD, or alternately, EB^2 : EC^2 :: AB.BD : AC.CD; whence (VI. 22) EB is to EC as a mean proportional between AB, BD to a mean between AC, CD; and since AB, BD, AC, CD

Find mean proportionals between AB, BD, and AC, CD; and on the base BC, describe (App. III. 6, cor.) the triangle EBC, having the angle BEC equal to the given angle, and BE to EC as the first mean to the second: E is the required point.

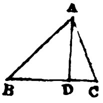


Join EA, ED, and about the triangle AED describe a circle, meeting EB, EC produced in F and G; and join FG. Then (const. and VI. 17, and 22) $BE^2 : CE^2 :: AB.BD : AC.CD$, or (III. 35, and alternately) $BE^2 : FB.BE :: CE^2 : GC.CE$. Hence (VI. 1) BE : BF :: CE : CG, and (VI. 2) BC, FG are parallel; wherefore (III. 26, cor.) the arcs AF, GD, and (III. 27) the angles AEF, DEG are equal.

Schol. The method of solution will be almost exactly the same, if the points A and D lie between B and C.

Prop. XXVIII. Theor.—The area of a triangle ABC is equal to half the continued product of two of its sides, AB, BC, and the sine of their contained angle B, to the radius 1.

Draw the perpendicular AD. Then (App. III. 1, cor.) AD = AB × sin B. Multiply by BC, and take half the product: and (I. 46, cor. 6) we have the area equal to ½AB×BC×sinB.



Cor. Hence (I. 34) the area of a parallelogram is equal to the continual product of two contiguous sides, and the sine of the contained angle.

Prop. XXIX. Theor.—Let a, b, c be the sides of a triangle, and s half their sum; the area is equal to the square root of the continual product of s, $s-a$, $s-b$, and $s-c$.

It was proved in the second corollary to the ninth proposition (page 168), that the sine of twice any angle is twice the product of the sine and cosine of the angle. Hence, by multiplying together the values of sin½A and cos½A, given in the corollaries to the sixth and seventh propositions, pages 166 and 167, and doubling the result, we get $\sin A = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$. Now, by the preceding proposition, the area of a triangle is found by multiplying the sine of one of its angles by the sides containing it, and taking half of the product: multiplying, therefore, the value now found for sinA, by bc, and taking half the product, we find the area to be $\sqrt{s(s-a)(s-b)(s-c.)}$

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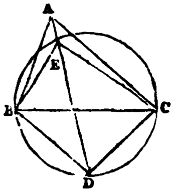
are given, these means are given. In the triangle EBC, therefore, the base, the ratio of the sides, and the vertical angle are given; whence, by the corollary to the sixth proposition of this book, the triangle, and consequently the point E are given.

Prop. XXX. Prob.—Given three points A, B, C, and the angles which straight lines drawn from them to a fourth point D, make with one another; to find the position of that point.



Join AB, BC, AC; and on BC describe the segment BDC, containing an angle equal to the one formed by straight lines drawn from B and C to the required point: then, if a segment were described on AB containing an angle equal to the one formed by lines drawn from the required point to A and B, the arcs of the segments would intersect one another in D, which is the required point. For if DA, DB, DC be joined, the angles BDC, BDA are obviously of the given magnitudes, as they are angles in segments described so as to contain those angles.

Method of Computation. Complete the circle BDCE, and let AD cut it in E; join BE, CE. Now, since the positions of the points A, B, C are given, the sides of the triangle ABC, and consequently its angles are given. Then, in the triangle BCE, the side BC is given, as are also the angles EBC, ECB, which (III. 21) are respectively equal to EDC, EDB. Hence BE may be computed; and then, in the triangle ABE, the sides AB, BE are given, and the contained angle ABE, which is the difference of ABC, EBC. The resolution of this triangle gives the angle BAE; and then, in the triangle ABD, the angles A and D, and the side AB are given; whence AD, BD can be computed; and CD may be found by means of the triangle ACD, or otherwise.

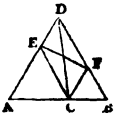
Schol. The solution is somewhat simplified, if A, B, C lie in a straight line: and still more, if they form a triangle, and D be in one of its sides. These cases present no difficulty.

It may be farther remarked, that if the circle BDCE pass through A, as it does when the angles BAC, BDC are supplementary, the problem is indeterminate or porismatic, since wherever the point D is taken in the arc BDC, the angles at D will be of the same magnitude.

Prop. XXXI. Prob.—To describe a parallelogram having its adjacent sides respectively equal to the segments of one straight line divided in extreme and mean ratio, and its diagonals to those of another divided in the same manner.

Divide any straight line AB in extreme and mean ratio in C, and on AB describe the equilateral triangle ABD; through C draw CE, CF parallel to DB, DA: CEDF is a parallelogram such as is required.

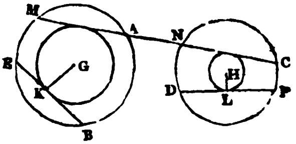
For the triangles ACE, BCF are obviously equilateral, and CE, CF are therefore equal to the segments into which AB is divided in C. Again, BA : AC :: AC : CB, or DA : AC :: DF : DE, because BA is equal to DA, AC to DF, and CB to DE. Hence (VI. 6) the triangles ADC, DFE are equiangular; and therefore (VI. 4, and alternately) ED or CB : AC :: EF : CD, and, by composition, AB : AC :: CD+EF : CD. But, from the last analogy, by inversion, AC : CB, or (const.) AB : AC :: CD : EF; whence (V. 11) CD+EF : CD :: CD : EF; and therefore (VI. def. 3) CD, EF are the segments of a line equal to their sum, cut in extreme and mean ratio.



PROP. XXXII. PROB.—To draw a straight line cutting two given circles, so that the part of it within one of them may be equal to a given straight line, and that it may divide the other into segments containing angles equal to given angles.

Let it be required to draw a straight line cutting the circles AB, CD, so that the part of it within AB may be equal to a given straight line, and that it may divide CD into segments containing angles equal to given angles.

In AB inscribe the chord BE equal to the given line, and (III. 34) draw the straight line DF dividing CD into segments containing angles equal to the given angles; from the centres G and H draw GK, HL perpendicular to the chords BE, DF, and with these perpendiculars as radii, and G, H as centres, describe circles; draw (App. I. 23) MANC a common tangent to these circles; it is the required line.



For (III. 14) MA, BE are equal, because they are equally distant from the centre; and therefore MA is equal to the given line. For the same reason NC is equal to DF, and therefore (III. 28) the arcs NC, DF are equal. But (III. 27) in the same circle, the angles which stand on equal arcs are equal: therefore the segments NDFC and DNCF contain equal angles; and, since (const.) DNCF contains an angle equal to one of the given angles, and since MA is equal to the given straight line, MC is the line required.

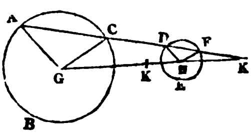
Schol. MC may evidently have as many different positions, as the common tangent to the interior circles. It is plain also, that the given straight line must not exceed the diameter of the circle in which the part of the intersecting line equal to it is to be.

PROP. XXXIII. PORISM.—If there be two given unequal circles, which have not the same centre two points may be

found, through either of which if any straight line whatever be drawn cutting them, it will divide them into segments similar to one another, each to each.

Let ABC, DEF be the circles, and G, H their centres; let the point K be so taken (VI. 10, and cor.) in the straight line passing through G and H, that its distances from those points may be proportional to the respective radii of the circles: any straight line drawn through K, and cutting the circles, cuts off similar segments.

Draw KFDCA cutting the circles, and join HF, HD; GC, GA. Then (const.) HK : HF :: GK : GC; wherefore (VI. 7) the triangles KHF, KGC having the angle K common, and the angles KHF, KGC acute, are equiangular; and (I. 28) HF, GC are parallel. In a similar manner it would be shown, that HD, GA are parallel. Hence (I. 34, cor. 4) the angles FHD, CGA are equal; and therefore the angles in the segments DEF, ABC being halves of these, are equal, and (III. def. 8) these segments are similar; as are also the remaining segments, since (I. 22, and ax. 3) the angles in them are equal.

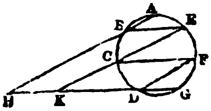


Schol. 1. It is plain, (App. I. 24) that when these points fall without the circles, the common tangents to the circles pass through them.

Schol. 2. If the circles be equal, there is one point which answers—that which bisects the distance between the centres. The other may be regarded as lying at an infinite distance: and any parallel to the line joining the centres, which cuts the circles, divides them into proportional segments.

Prop. XXXIV. Theor.—**Let A be a point in the circumference of the circle ADG, and AB, BC, CD any number of equal arcs, and AE, EF, FG as many others equal to them, and all taken together less in amount than the circumference: then, if the chords AB, BE, CF, DG be drawn, and if AB, DG be produced to meet in H; GH is equal to the sum of the chords BE, CF, DG.**

Draw ECK and FD. Then, because the arcs AB, BC, AE, &c. are equal, BE, CF, DG are parallel to one another; and so also are AH, EK, FD. Hence EH, FK are parallelograms, and HK, KD are equal to BE, CF. Add DG; then, HG is equal to BE, CF, DG.

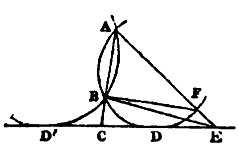


Prop. XXXV. Prob.—From two given points to draw two straight lines meeting in a given straight line, and forming with one another the greatest angle possible.

From two given points A and B, let it be required to draw two straight lines meeting in a given straight line CD, and making with one another the greatest angle possible.

Through the given points describe (App. II. 1) a circle touching CD in D: D is the required point.

In CD take any point E on either side of D, and draw EB, EFA, and FB. Then (I. 16) the exterior angle AFB is greater than AEB; and AFB is equal (III. 21) to the angle contained by straight lines drawn from D to A and B; the last angle, therefore, is also greater than AEB: that is, than any angle formed by straight lines drawn from A and B, to any point in CD except D.



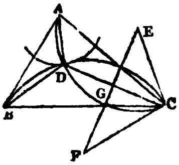
Schol. 1. If A and B be at unequal distances from CD, there may be two circles touching the given line in D and D′; and the point D gives the maximum angle of those on one side of the straight line ABC, while D′ is the vertex of the maximum angle among those on the other side of ABC. This will receive illustration from supposing straight lines drawn from A and B to a point varying its position along EC; since the angle contained by those lines will continually diminish, as the point moves from D to C, where it vanishes. The point then continuing to move on towards D′, the angle gradually increases till the point arrives at D′, when the angle is again a maximum, and beyond that point it continually diminishes.

Schol. 2. Of the angles formed by straight lines drawn from two given points to meet in the circumference of a given circle, those are the greatest or least possible which have their vertices at a point of contact of a circle passing through the given points, and touching the given circle.

Prop. XXXVI. Prob.—In a triangle ABC, each angle of which is less than a right angle and a third, to find a point such that the sum of the straight lines drawn from it to the three angles may be a minimum.

On two of the sides BC, CA, describe (III. 33) segments of circles containing each an angle equal to a right angle and a third; from D, their point of intersection, draw DA, DB, DC: these are the lines required.

Since ADC, BDC are each a right angle and a third, and the three angles at D are equal to four right angles, ADB must also be of the same magnitude. Now, if AD be supposed to be of a fixed magnitude, and with it as radius, and A as centre, a circle be described, the radius AD makes equal angles with DB, DC; and, therefore (App. I. 13. cor.) the sum of BD, DC is a minimum. Hence it is plain, that the sum of the three lines cannot be a minimum, unless the angles ADB, ADC be equal. In the same manner it would be proved, that if BD be assumed of a fixed magnitude, the sum of AD, DC is a minimum; and that the sum of AD, BD, CD cannot be a minimum, unless BD make equal angles with DA, DC; that is, from what has been already proved, unless the three angles at D be equal. But they are so by construction; and therefore D is the required point.

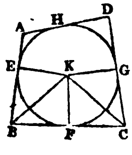


Method of Computation. It is plain that the centres of the arcs ADC, BDC, are the centres of equilateral triangles described externally on AC, BC; that is, they are the points K, H in the figure for the 26th proposition of the first book of this Appendix. Now by the demonstration of that proposition, the angle HKC is equal to FAC: and, since the sides FC, CA, and the contained angle ACF ($=ACB+60^\circ$) are given, the angle CAF or CKH, or in this proposition CEF, can be found by the second case of trigonometry. Then, since ACE is 30°, if a perpendicular be drawn from E to AC, EC can be computed; and thence in the right-angled triangle EGC, CG, which is half of CD, may be found.

Prop. XXXVII. Theor.—Two opposite sides of a quadrilateral described about a circle are together equal to the other two.

Let ABCD be a quadrilateral, the sides of which touch the circle EG in E, F, G, H; the sum of AB, DC is equal to the sum of AD, BC.

For (III. 17, schol.) AE is equal to AH, and BE to BF; whence, by addition, AB is equal to the sum of AH, BF. It would be proved in a similar manner, that DC is equal to the sum of HD, FC; whence again by addition, AB, DC are together equal to AD, BC.



Schol. The converse of this is also true. For, let the sum of AD, BC be equal to that of AB, DC, and (IV. 4) describe a circle touching any three sides AB, AD, DC, in E, H, G; join K, the centre of the circle, with E, B, C, G; draw also KF perpendicular to BC. Then KF is equal to the radius of EG. For since the hypotenuse KB is common to the two right-angled triangles KEB, KFB, and KC to KGC, KFC, if KF were less than KE or KG, BF and FC, that is BC, would (I. 47, cor. 6) be greater than BE and CG; while if KF were greater than KE or KG, BC would be less than BE and CG: but BC is equal to BE, CG, because AE, DG are equal to AD. Therefore F is on the circumference; and BC touches the circle, because the angles at F are right angles.

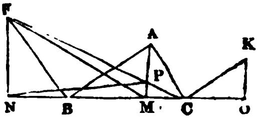
It is plain from this scholium, and from the fourth proposition of the first book of this Appendix, that a quadrilateral which has two opposite sides equal to the other two, and the opposite angles equal to two right angles, may have one circle inscribed in it, and another described about it. It also follows, that any quadrilateral having two opposite sides equal to the other two, may have a circle described touching all its sides, however the angles may vary in their magnitude, and even should one of them become re-entrant.

PROP. XXXVIII. PROB.—The circumference of a circle being divided into any number of equal parts, it is required to describe as many circles as there are parts, each touching the given circle and two of the others.

This will be effected simply by drawing tangents to the circle through the points of division; by drawing lines from their points of intersection to the centre; and by inscribing circles in the triangles thus formed. Other circles, touching the given one externally, and each touching two others, will be found by describing circles touching the sides of the circumscribed figure, and the radii produced through the intersections of the tangents. The proof is easy in both cases.

PROP. XXXIX. THEOR.—In the figure for the forty-seventh proposition of the first book of Euclid, the straight lines BK, CF, and the perpendicular from the right angle to the hypotenuse, pass through the same point.

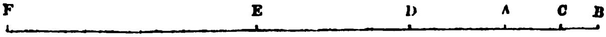
Produce BC both ways, and draw AM, FN, KO perpendicular to it. Let AM, FC intersect each other in P, and join PN, FM. Then, since ABF is a right angle, ABM, NBF are together equal to a right angle, and therefore to NBF, BFN; whence ABM, BFN are equal; and (I. 26) since BF, BA are equal, BN is equal to AM, and FN to BM. In the same manner it would be proved, that CO, OK are respectively equal to AM, MC; whence NB is equal to CO, and NC to BO. Now (I. 37) the triangles NMP, FMP are equal: to each add PMC, and the wholes NPC, FMC are equal. So likewise are their doubles, which are the rectangles NC.PM, and MC.FN or MC.BM. It would be proved in a similar manner, that the rectangle under BO or NC, and the part of NA intercepted between M and the



straight line joining BK, is likewise equal to BM.MC: and therefore, FC, BK cut AM in the same point.*

Prop. XL. Prob.—Let a given straight line FE be cut in extreme and mean ratio, and in its continuation let ED be taken equal to its greater segment; in like manner let DA be equal to the greater segment of ED, AC equal to the greater segment of DA, and so on: it is required to find the length of the entire line that would be obtained by the addition of an infinite number of these successive lines, FE, ED, DA, &c.

Let S be put to denote the required sum. Then, by the corollary to the 19th proposition of the Supplement to the fifth book, FE—ED : FE :: FE : S. But (II. 11, schol.) FE—ED : ED :: ED : EF, or as EF : FD; whence, alternately, FE—ED : EF :: ED : FD; and therefore (V. 11) ED : FD :: EF : S, or,



alternately, ED : EF :: FD : S. But ED : EF :: FD : FD+FE, and therefore FD : FD+FE :: FD : S, and (V. 9) S=FD+FE=2FE+ED; that is, the required sum is equal to twice the given line together with its greater segment; or to a straight line, of which, when cut in extreme and mean ratio, the given line is the less segment.

Prop. XLI. Prob.—In a given semicircle ACB let another be inscribed lying in the opposite direction; in the second let another be inscribed in a similar manner; in it another, and so on; their diameters being all parallel: it is required to find the position of the point to which the centres approach, when the process is continued without limit.

Through the centre F draw FC perpendicular to AB; then, if radii be drawn bisecting the right angles at F, and if their extremities D, E be joined by DGE, it is evident that the triangles DFE, FGD, FGE are all right-angled and isosceles; and therefore G is the centre of the first inscribed semicircle. Let this be described; and then by bisecting the angles at G, the next semicircle HGK and its centre L may be determined; and so on. Join CD, HF. Then the triangles CGD, HLF, have each a right angle, and the angles GDC, LHF are equal, because they stand on like parts of the circumferences of the two circles, of which ACB, DFE are halves. Hence (VI. 4, and alternately) CG : LF : : GD or GH : LH ; that is, as the diagonal of a square to its side. In exactly the same manner, if M and N, the centres of the next semicircles were found, it would be shown, that GM is to LN, as the diagonal of a square to its side; and the same must hold in regard to each successive pair ot semicircles. Now, the square of HG or DG is double of the square of GL, and the square of DF or CF is double of the square of DG, and therefore quadruple of the square of GL; wherefore (II. 4, cor. 2) CF is double of GL, and consequently of the sum of CG, LF: and in the same manner, GL would be shown to be double of the sum of GM, LN, and so on. Hence (XII. lem. 1) if from CF there be taken CG, LF, and if from GL there be taken GM, NL, and so on ; a remainder will at length be obtained less than any line however small: and therefore the two lines made up of CG, GM, &c. and of FL, LN, &c. may be made to approach a point in FC, so that the distances between it and their extremities will be as small as we please. Now (V. 12) the sum of CG, GM, &c. is to the sum of FL, LN, &c. as CG to FL, that is, as the diagonal of a square to the side; and therefore, if CF be divided in O, so that CO is to OF as the diagonal of a square to the side, O will be the required point.

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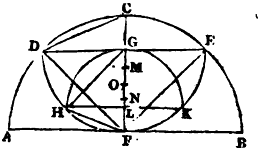
* This is one out of many curious propositions that may be founded on the figure for Euclid's proof of the 47th proposition of the first book. Another very simple one is, that if GH, FD, EK be joined, each of the triangles thus formed is equal to ABC. This follows immediately from the second corollary to the 38th proposition of the first book. Thus, the triangles FBD, ABC, are equal, because FB, BD are respectively equal to AB, BC, and the contained angles supplemental.

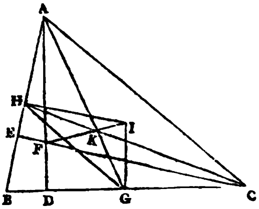


PROP. XLII. PROB.—A semicircle ACB being given, and other semicircles being described as in the preceding proposition; it is required to find the sum of the areas of all those inscribed semicircles.

Circles (XII. 2), and consequently semicircles, are as the squares of their diameters or of their radii. Now the square of GD is half the square of DF or CF, and therefore the semicircle DFE is half of ACB. For the same reason HGK is half of DFE; and universally, each semicircle is half of the one in which it is inscribed. Hence the entire amount will be the sum of the infinite series $\frac{1}{2}$ACB + $\frac{1}{4}$ACB + $\frac{1}{8}$ACB + $\frac{1}{16}$ACB + &c.; and therefore (V. SUP. 19) $\frac{1}{2}$ACB — $\frac{1}{4}$ACB : $\frac{1}{2}$ACB : : $\frac{1}{2}$ACB : ACB, the required sum; and it thus appears that the sum of all the inscribed semicircles is equal to the given semicircle.

PROP. XLIII. THEOR.—In any triangle, the centre of the circumscribed circle, the point in which the three perpendiculars intersect one another, and the point of intersection of the straight lines drawn from the angles to bisect the opposite sides, lie all in the same straight line.

Let ABC be a triangle, and let the two perpendiculars AD, CE intersect in F; bisect AB, BC in H, G, and draw AG, CH intersecting in K; draw also GI, HI perpendicular to BC, BA, and intersecting in I. Then (VI. H) F is the intersection of the three perpendiculars, K (App. I. 7) the intersection of the three lines drawn from the angles to bisect the opposite sides, and (IV. 5, cor. 1) I is the centre of the circumscribed circle. Join FK, KI: FKI is a straight line.

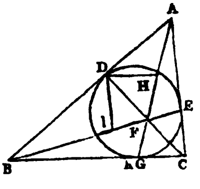


Join HG: it is (VI. 2, and 4) parallel to AC, and is half of it. Also the triangles ACF, GHI are (I. 34, cor. 4) equiangular, and therefore GI is half of AF. So likewise (App. I. 7, schol.) is KG of KA. Hence the two triangles AKF, GKI have the alternate angles KAF, KGI equal, and the sides about them proportional: therefore (VI. 6) the angles AKF, GKI are equal, and (I. 14, cor.) since AKG is a straight line, FKI is also a straight line.

Schol. It is plain (VI. 4) that FK is double of KI. We have also seen that AF is double of GI. Hence it appears, that the distance between any of the angles and the point of intersection of the three perpendiculars, is double of the perpendicular drawn from the centre of the circumscribed circle to the side opposite to that angle.

Prop. XLIV. Theor.—**Straight lines drawn from the angles of a triangle to the points in which the opposite sides touch the inscribed circle, all pass through the same point.**

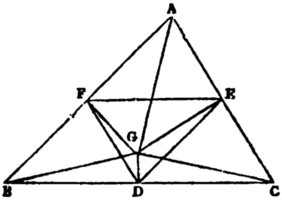
Let ABC be a triangle, and D, E the points in which the sides AB, AC touch the inscribed circle; draw BFE, CFD; draw also AFG cutting BC in G: G is the point in which BC touches the inscribed circle.



If possible, let another point K be the point of contact, and draw DH, DI parallel to BC, CA. Then in the similar triangles FDI, FCE, FD : DI :: FC : CE, or CK; and in the similar triangles FDH, FGC, DH : DF :: GC : FC; from which and from the preceding analogy we get, *ex æquo*, DH : DI :: CG : CK. Again BD : DI :: BA : AE or AD :: BG : DH. Hence, alternately, and by inversion, BG : BD :: DH : DI; whence (V. 11) BG : BD or BK :: CG : CK, or alternately, BG : CG :: BK : CK; and by composition, BC : CG :: BC : CK; and therefore CG, CK are equal; that is, G and K coincide, and AFG passes through the point in which BC touches the circle.

PROP. XLV. THEOR.—In a triangle, the sum of the perpendiculars drawn from the centre of the circumscribed circle to the three sides, is equal to the sum of the radii of the inscribed and circumscribed circles.

Let ABC be a triangle, having its sides bisected in D, E, F, by perpendiculars meeting in G, the centre of the circumscribed circle: the sum of GD, GE, GF is equal to the sum of the radii of the inscribed and circumscribed circles.



Join GA, GB, GC, and DE, EF, FD. Then, putting a, b, c to denote the sides opposite to the angles A, B, C, we have (VI. 2, and 4) $FE = \frac{1}{2}a$, $FD = \frac{1}{2}b$, and $DE = \frac{1}{2}c$; and (APP. I. 4) since AEG, AFG are right angles, a circle may be described about the quadrilateral AEGF. For a like reason circles may be described about BDGF and CDGE. Hence (VI. E) $FE.AG = AF.GE + AE.FG$; or, by doubling, $a.AG = c.GE + b.FG$. In the same manner, it would be shown, since AG, BG, CG are equal, that $b.AG = c.GD + a.FG$, and $c.AG = a.GE + b.GD$. Hence, by addition, $(a+b+c)\ AG = (a+c)\ GE + (a+b)\ GF + (b+c)\ GD$. Now $b.GE$ is evidently equal to twice the triangle AGC, $c.GF$ equal to twice AGB, and $a.GD$ equal to twice BGC; also, denoting the radius of the inscribed circle by r, we have (IV. 4, cor. 6) $r\ (a+b+c)$ equal to twice the area of the triangle ABC, and consequently $r\ (a+b+c) = b.GE + c.GF + a.GD$. Hence, by addition, $(a+b+c)\ AG + r\,(a+b+c) = (a+b+c)\ GE + (a+b+c)\ GF + (a+b+c) GD$; and consequently $AG + r = GE + GF + GD$.

Cor. Since, by the scholium to proposition XLIII. of this book, the parts of the three perpendiculars of the triangle, between their common intersection aud the three angles, are respectively double of GD, GE, GF, the sum of those parts of the perpendiculars is equal to the sum of the diameters of the inscribed and circumscribed circles.

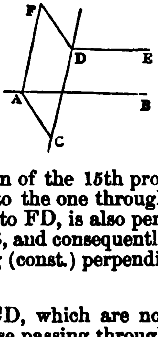
BOOK IV.

PROPOSITIONS.

Prop. I. Prob.—To draw a straight line perpendicular to two given straight lines, not in the same plane.

Let AB, CD be two given straight lines not in the same plane; it is required to draw a straight line which will be perpendicular to them both.

Through any point D in one of them, draw (I. 31) DE parallel to the other, and (XI. 12) through the same point D draw DF perpendicular to the plane passing through DE, DC: let the plane passing through CD, DF meet AB in A, and (I. 12) from A draw AC perpendicular to CD; AC is the perpendicular required.

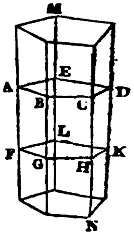


Draw AF parallel to CD. Then (I. 28) CA, DF are parallel, because they are both perpendicular to CD. FA, AB, therefore, being parallel to CD, DE, the plane passing through FA, AB is parallel (XI. 15) to the one passing through CD, DE; and therefore FD, being (const.) perpendicular to the plane through CD, DE, is also, by the demonstration of the 15th proposition of the eleventh book, perpendicular to the one through FA, AB; and therefore AC, which is parallel to FD, is also perpendicular (XI. 8) to the plane through FA, AB, and consequently (XI. def. 1) to the straight line AB: and being (const.) perpendicular to CD, it is the line required.

Schol. Through the two straight lines AB, CD, which are not parallel, two parallel planes can be drawn—those passing through AB, AF, and through CD, DE: but no other parallel planes besides these can be drawn through AB, CD, as AB can meet the plane CF in only one point. For the same reason, there is only one common perpendicular to AB, CD. On the contrary, were AB, CD parallel, there might be innumerable parallel planes, and common perpendiculars. It is also evident, since AC is the perpendicular distance between the two planes, that it is less than any line between them which is not perpendicular to both, and therefore that it is the least straight line that can be drawn between AB and CD.

PROP. II. THEOR.—The sections of a prism by parallel planes, are equal and similar figures.

Let the prism MN be cut by the two parallel planes ABCDE, FGHKL; their sections with it are equal and similar figures.



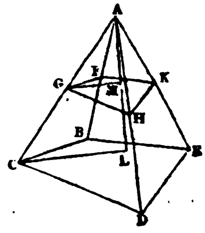
For (XI. 16) AB, FG are parallel, and (I. 34) they are equal. For the same reason BC, GH are equal and parallel; as are also CD, HK; DE, KL; and AE, FL. The figures AD, FK, therefore, have their sides equal, each to each. Their several angles are also (XI. 10) equal; for they are contained by straight lines which are parallel; and therefore the figures are equal and similar.

Cor. A section of a prism by a plane parallel to the base, is equal and similar to the base.

Schol. Since (XI. def. 24) a cylinder is described by the revolution of a rectangle about one of its sides, it is plain that any straight line in the rectangle perpendicular to the fixed line, will describe a circle parallel to the base; and hence every section of a cylinder by a plane parallel to the base, is a circle equal to the base.

PROP. III. THEOR.—The section of a pyramid by a plane parallel to its base, is a figure similar to the base.

Let the pyramid ABCDE be cut by the plane FGHK parallel to its base BCDE; the section FGHK is similar to the base.



It would be proved as in the preceding proposition, that the figures GK, CE are equiangular. Then, because the triangles AFG, ABC are equiangular, FG : BC :: AG : AC; and because AGH, ACD are equiangular, GH : CD :: AG : AC. Hence (V. 11, and alternately) FG : GH :: BC : CD; and in a similar manner it would be proved, that the sides about the other angles of GK, CE are proportionals. Hence (VI. def. 1) GK is similar to CE.

Cor. Draw AML perpendicular to CE, and consequently to GK: the altitude AL, and the straight lines AB, AC, AD, AE are cut proportionally by GK. Join CL, GM: these (XI. 2, and 16) are in the same plane, and are parallel to one another. Hence (VI. 2) AF : FB :: AG : GC :: AM : ML, &c.

Schol. 1. Since (XI. def. 21) a cone is described by the revolution of a right-angled triangle about one of its legs, it is plain that any straight line in the triangle perpendicular to the fixed leg, will describe a circle parallel to the base; and the radius of that circle will be to the radius of the base, as the altitude of the cone cut off to that of the whole cone.

Schol. 2. The pyramid or cone cut off from another pyramid or cone, by a plane parallel to the base, is similar (XI. def. 8, and 27) to the whole pyramid or cone.

Prop. IV. Theor. — If the altitude of a parallelepiped, and the length and perpendicular breadth of its base be all divided into parts equal to one another, the continued product of the number of parts in the three lines is the number of cubes contained in the parallelepiped, each cube having the side of its base equal to one of the parts.

First, suppose the parallelepiped to be rectangular. Then planes parallel to the base passing through the points of section of the altitude, will evidently divide the solid into as many equal solids, as there are parts in the altitude; and each of these partial solids will be composed of as many cubes, as the base contains squares each equal to a base of one of the cubes. But (I. 46, cor. 4) the number of these squares is the product of the length and breadth of the base; and hence the entire number of cubes will be equal to the product of the three dimensions, the length, breadth, and altitude.

If the base be not rectangular, its area (I. 46, cor. 5) will be the product of its length and perpendicular breadth: and it is evident that the product of this by the altitude, will be the number of cubes as before.

Lastly, if the insisting lines be not perpendicular to the base, still the oblique parallelepiped is equal (XI. 29, and 30) to a rectangular one of the same altitude; and therefore the number of cubes will be found as before, by multiplying the area of the base by the altitude.

Cor. 1. Hence it is evident that the *volume*, or *numerical solid content*, or, as it is also called, the *solidity*, of a parallelepiped, is the product of its altitude, and the area of its base, both expressed in numbers: and it is plain, that the same holds in regard to any prism whatever, and also in regard to cylinders.*

Cor. 2. The content of a pyramid or cone is found by multiplying the area of the base by the altitude, and taking a third of the product. For (XII. 7, cor. 1) a pyramid is a third part of a prism, and (XII. 10) a cone a third part of a cylinder, of the same base and altitude.

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* As an example, suppose the length, breadth, and depth of a rectangular chest to be respectively 4 feet, 3 feet, and 2 feet: its content will be $4 \times 3 \times 2 = 24$ cubic feet; that is, 24 cubes having their bases each a square foot. The measuring unit is here a cubic foot; but it is evident, that it may, according to convenience, be a cubic inch, a cubic yard, &c.

As another example, let the radius of the base of a cylinder be 10 inches, and its altitude 15 inches. Then (App. I. 39, cor. 2) the area of the base is 314·1593; the product of which by 15 is 4712·3895 cubic inches, the content of the cylinder.

Cor. 3. The content of a polyhedron may be found by dividing it into pyramids, and adding together their contents. The division into pyramids may be made either by planes passing through the vertex of one of the solid angles, or by planes passing through a point within the body.

PROP. V. THEOR. — The surfaces of two similar polyhedrons may be divided into the same number of similar triangles, similarly situated.

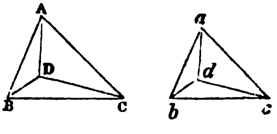
This follows immediately from the definition (XI. def. 8) of similar bodies bounded by planes, if the sides or faces of the polyhedron be triangles: and any face in the one, and the corresponding face in the other, which are not triangles, are yet similar, and may be divided (VI. 20) into the same number of similar triangles, similarly situated.

Cor. Hence it would be shown, as in the 20th proposition of the sixth book, that the surfaces of the polyhedrons are proportional to any two of their similar triangles; and therefore (VI. 19) that they are to one another in the duplicate ratio of the homologous sides of those triangles, that is, of the edges or intersections of the similar planes. Hence also the surfaces are proportional (VI. 20. cor. 3) to the squares of the edges.

PROP. VI. THEOR.—Triangular pyramids are similar, if two faces in one of them be similar to two faces in the other, each to each, and their inclinations equal.

Let ABC, *abc* be the bases, and D, *d* the vertices of two triangular pyramids, in which ABC, DBC are respectively similar to *abc*, *dbc*, and the inclination of ABC, DBC equal to that of *abc*, *dbc*: the pyramids are similar.

To demonstrate this, it is sufficient to show that the triangles ABD, ACD are similar to *abd*, *acd*, for then the solid angles (XI. B) will be equal, each to each, and (XI. def. 8) the pyramids similar. Now the angles ABD, *abd* are equal. For if they be equal, since the other plane angles at B and *b* are equal, the inclinations of ABC, DBC, and of *abc*, *dbc*, are (XI. A) equal. But if ABD be not equal to *abd*, conceive DBC to revolve about BC till ABD become equal to *abd*: then (XI. A) the inclination of ABC, DBC would still be equal to that of *abc*, *dbc*, which is absurd; since the inclinations of these planes, at first (hyp.) equal, are rendered unequal by the motion of DBC: therefore ABD, *abd* are equal. Then (hyp.) DB : BC : : *db* : *bc*, and BC : BA : : *bc* : *ba*; whence, *ex æquo*, DB : BA : : *db* : *ba*; and therefore (VI. 6) the triangles ABD, *abd* are equiangular, and conse-



quently similar: and it would be proved in the same manner, that ACD, *acd* are similar. Therefore (XI. def. 8) the pyramids are similar.

PROP. VII. THEOR.—Triangular pyramids are similar, if three faces of one of them be respectively similar to three faces of the other.

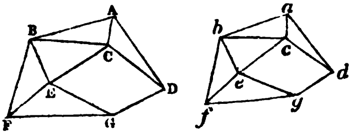
In the triangular pyramids ABCD, *abcd* (see the preceding figure) let the faces ABC, ABD, DBC be similar to *abc, abd, dbc*, each to each: the pyramids are similar.

For (VI. def. 1) AD : DB :: *ad* : *db*, and DB : DC :: *db* : *dc*; whence, *ex æquo*, AD : DC :: *ad* : *dc*. Also DC : CB :: *dc* : *cb*, and CB : CA :: *cb* : *ca*; whence, *ex æquo*, DC : CA :: *dc* : *ca*; and therefore (VI. 5) the triangles ADC, *adc* are equiangular, and (XI. B and def. 8) the pyramids are similar.

PROP. VIII. THEOR.—Similar polyhedrons may be divided into the same number of triangular pyramids, similar, each to each, and similarly situated.

Let ABCDEFG and *abcdefg* be similar polyhedrons, having the solid angles equal which are marked with the corresponding large and small letters: they may be divided into the same number of similar triangular pyramids similarly situated.

The surfaces of the polygons may be divided (APP. IV. 5) into the same number of similar triangles, similarly situated: then planes passing through any two corresponding solid angles, A, *a*, and through the sides of all these triangles, except those forming the solid angles, A, *a*, will divide the polyhedrons into triangular pyramids, similar to one another, and similarly situated.



The pyramids thus formed have each one solid angle at the common vertex A or *a*: and these solid angles may be of three classes; 1*st*, those which have two of their faces coinciding with faces of one of the polyhedrons; 2*d*, those which have only one face coinciding; and 3*d*, those which lie wholly within the solid angle A or *a*. Now those of the first kind in one of the polyhedrons, are similar to the corresponding ones in the other, by the seventh proposition of this book; and those of the second kind by the sixth. From the polyhedrons take two of these similar pyramids, and the remaining bodies will be similar: as the boundaries common to them and the pyramids are (APP. V. 6, or 7) similar triangles; and their other boundaries are similar, being faces of the proposed polyhedrons. Also the solid angles of the remaining bodies are equal, as some of them are angles of the primitive polyhedrons, and the rest are either trihedral angles which are contained by equal plane angles, or may be divided into such. From these remaining bodies other similar triangular pyramids may be taken in a similar manner; and the process may be continued till only two similar triangular pyramids remain; and thus the polyhedrons are resolved into the same number of similar triangular pyramids.

PROP. IX. THEOR.—Similar polyhedrons are to one another in the triplicate ratio of their homologous sides.

For, by the preceding proposition, they may be divided into the same number of similar triangular pyramids: and these pyramids (XII. 8) are to one another in the triplicate ratio of their homologous sides. But because the polyhedrons are similar, their homologous sides and diagonals are all in the same ratio; and, *ex æquo*, ratios which are triplicate of equal ratios are equal. Hence, as any of the pyramids in one of the polyhedrons, to the similar one in the other, so is any other in the first to the similar one in the other; and therefore (V. 12) the one polyhedron is to the other, as any pyramid in the first, to the similar one in the other; that is, in the triplicate ratio of their homologous sides or edges.

Cor. 1. Since cubes are similar bodies, they have to one another the triplicate ratio of that which their sides have; and therefore similar polyhedrons are to one another as the cubes of their homologous sides.

Cor. 2. If four straight lines be continual proportionals, the first is to the fourth as any polyhedron described on the first, is to a similar polyhedron, similarly described on the second.

PROP. X. THEOR.—A section of a sphere by a plane is a circle.

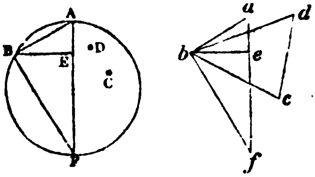
Since the radii of the sphere are all equal, each of them being equal to the radius of the describing semicircle, it is plain that if the section pass through the centre, it is a circle of the same radius as the sphere. But if the plane do not pass through the centre, draw (XI. 11) a perpendicular to it from the centre, and draw any number of radii of the sphere to the intersection of its surface with the plane. These radii, which are equal, are the hypotenuses of right-angled triangles, which have the perpendicular from the centre as a common leg; and therefore (I. 47, cor. 5) their other legs are all equal: wherefore the section of the sphere by the plane is a circle, the centre of which is the point in which the perpendicular cuts the plane.

Schol. 1. All the sections through the centre are equal to one another, and are greater than the others. The former are therefore called *great circles*, the latter *small* or *less circles*.

Schol. 2. A straight line drawn through the centre of a circle of the sphere perpendicular to its plane, is a diameter of the sphere. The extremities of this diameter are called the *poles* of the circle. It is plain (I. 47, cor. 5) that chords drawn in the sphere from either pole of a circle to the circumference, are all equal; and therefore (III. 28) that arcs of great circles between the pole and circumference are likewise equal.

Prop. XI. Prob.—To find the diameter of a given sphere.

Let A be any point in the surface of the given sphere, and take any three points B, C, D at equal distances from A. Describe the triangle *bcd* having *bc* equal to the distance or chord BC, *cd* equal to CD, and *bd* to BD. Find *e* the centre of the circle described about *bcd*, and join *be*: draw *aef* perpendicular to *be*, and make *ba* equal to BA; draw *bf* perpendicular to *ba*, and *af* is equal to the diameter of the sphere.



Conceive a circle to be described through BCD, and E to be its centre; that circle will evidently be the section of the sphere by a plane through B, C, D; and it will be equal to the circle described about *bcd*. Conceive the diameter AEF to be drawn, and BA, BE, BF to be joined. Then, in the right-angled triangles ABE, *abe*, the sides AB, BE are respectively equal to *ab*, *be*, and therefore (I. 47, cor. 5) the angles A, *a* are equal. Again, in the right-angled triangles ABF, *abf*, the angles A, *a* are equal, and also the sides AB, *ab*: hence (I. 26) the sides AF, *af* are equal; that is, *af* is equal to the diameter of the sphere.

NOTES AND ILLUSTRATIONS.

ELEMENTS, BOOK I.

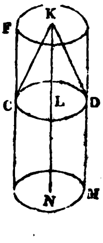
DEFINITIONS.

To what is stated in the foot notes, it may here be added, that several changes have been made in the arrangement of the definitions. The definition of parallel straight lines is placed after the definitions of an angle and its species; the inclination of lines and their parallelism equally depending on their mutual positions. The definitions of rectilineal figures are made to precede the definition of the circle; and a definition of the parallelogram is given, and is placed before those of its species, the rectangle, the square, the rhombus, and the rhomboid.

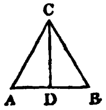
Euclid has defined a straight line to be *that which lies evenly, or equally, between its extreme points*. This definition neither gives a clearer idea of a straight line, than every one has without a definition, nor does it afford any assistance in arriving at the properties of straight lines; and therefore Playfair's definition has been adopted in this edition. From him also the definition of a point has been taken. In Dr. Simson's edition, a point is defined to be "that which has no parts, or no magnitude." This is objectionable, as being wholly negative.

The following illustrations, taken in substance from Dr. Simson, are well deserving of attention.

It is necessary to consider a solid, that is, a magnitude which has length, breadth, and thickness, for the purpose of understanding aright the definitions of a surface, line, and point, for these all arise from a solid, and exist in it. The boundary, or boundaries, which contain a solid, are surfaces; or the boundary which is common to two contiguous solids, or which divides one solid into two contiguous parts, is a surface. Thus, if CD be a boundary of the solid CM, or the common boundary of the two solids CM, FD, it is called a surface; and it is in the one solid as well as in the other, and has no thickness. For if it have any, this thickness must be either a part of the thickness of the solid CM, or FD, or of the thickness of both. It cannot be a part of the thickness of CM; for if this solid be removed from FD, the surface CD, the boundary of DF, remains as it was: and it would appear in a similar manner, that its thickness can be no part of the solid FD. The surface CD, therefore has only length and breadth, without any thickness.



The boundary, again, of a surface is a line; or a line is the common boundary of two contiguous surfaces, or it is that which divides one surface into two contiguous parts. Thus, if CD be a boundary of the surface ACD, or of the contiguous surfaces ACD, BCD, it is called a line, and it has no breadth. For if it have any, this must be part of the breadth, either of the surface ACD, or BCD, or of both. It is not a part of the breadth of ACD; for if this surface be removed from BCD, the line CD, which is a boundary of BCD, remains the same as it was. In like manner, it would be shown, that the supposed breadth is not a part of BCD. Therefore the line CD has no breadth; neither has it any thickness, because the surface in which it is, has no thickness. A line, therefore, has only length, without breadth or thickness.



Lastly, the boundary of a line is called a point: or a point is the common boundary or extremity of two contiguous lines. Thus, in the last figure, D, one extremity of the line AD, or the common extremity of the two lines AD, BD, is a point; and it has no thickness or breadth, because it is in a line, and a line has neither thickness nor breadth. Neither has the point D any length. For if it had any, this would be a part of the length, either of AD, or BD, or of both. It is not a part of the length of AD; for if AD were removed, D would still remain as before, the boundary of DB; and it would be shown in a similar manner, that its length is not a part of BD. Hence, a point has a certain position, but it has not length, breadth, or thickness.

GEOMETRY has been defined to be that science which treats of the properties of extension, and consequently of the properties of solids, surfaces, and lines. It forms the subject of this volume, with the exception of Euclid's fifth book and the Supplement to it, which are not confined to the consideration of geometrical magnitudes, but are equally applicable in arithmetic.

Plane geometry investigates the properties of lines and figures lying in the same plane; while *solid geometry* treats of solids, and of lines lying in different planes. The first, second, third, fourth, and sixth books of Euclid belong exclusively to the former: the eleventh and twelfth chiefly to the latter.

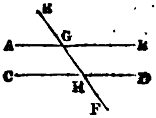
Elementary geometry treats of the straight line and circle, of solid figures bounded by planes, and of the "three round bodies,"—the cylinder, the cone, and the sphere. The *higher geometry* treats of the conic sections and other curves, and of the bodies produced by their rotations round fixed straight lines, or otherwise dependent on those curves.

PROP. VIII. AND XXVI.—In addition to what is demonstrated by Euclid in these two propositions, it is proved in each, in this edition, that the areas of the triangles are equal. This shortens the demonstrations of several subsequent propositions.

PROP. XXIX.—No subject in elementary geometry has so much perplexed mathematicians, both ancient and modern, as the theory of parallel lines. There is no difficulty in showing, that if straight lines in the same plane make certain angles equal, the lines will never meet, and are therefore parallel, as is done in the 27th and 28th propositions of this book; but to establish the converse, by showing, that if straight lines be parallel, they make certain angles equal, which is the object of the 29th proposition, has never been effected in a manner perfectly unobjectionable. The truth of the conclusions is universally admitted; but to demonstrate these conclusions with that rigour, of which geometrical reasoning is, in other cases, susceptible, has been found to present difficulties of no ordinary kind. Euclid has been unable to effect this by means of his definition of parallels, and has been obliged to assume, in his 12th axiom, a principle, which, though readily admitted after the 28th proposition is understood, is by no means self-evident.

To obviate these objections, some writers have endeavoured to prove the 12th axiom and the 29th proposition by means of the properties of lines established by Euclid, in what precedes that proposition; others have assumed some other axiom, supposed to be more obvious than Euclid's; and some have given a new definition of parallels. A short statement and examination of a few of the principal attempts of this kind, will make the student aware of the difficulty of the subject, and will show the necessity of great care and caution in all such inquiries.

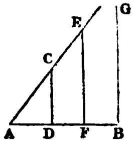
1. The earliest attempt with which we are acquainted, to establish the theory of parallels, without a new definition or axiom, is that of Ptolemy, the celebrated author of the astronomical system which is called after his name. His proof, as given by Proclus in the fourth book of his Commentaries, is in substance as follows. Let the straight line EGHF fall on the two parallels AB, CD; the angles BGH, DHG are together equal to two right angles. For, if not, let them be greater. Then, because the lines GB, HD are not more parallel than GA, HC, the angles AGH, CHG are also greater than two right angles; and therefore the four interior angles at G and H, are together greater than four right angles, which (I. 13) is absurd. Therefore BGH, DHG are not greater than two right angles; and it may be shown in a similar manner, that they are not less: they are therefore equal to two right angles.



Now, this cannot be regarded as a proof arising from the definition of parallel lines. By that definition, AB, CD never meet; but we cannot hence infer immediately, that the interior angles on one side of GH are greater or less than two right angles, according as those on the other side are: and though we readily admit that this is likely to be true, it is from the loose idea which we have concerning parallel lines, as lying in the same direction, and not from their definition. It is therefore to be regarded as a disguised axiom, independent of the definition; or, at best, as an illustration, showing the reasonableness of the last part of the 29th proposition.

2. In 1787, an attempt to prove the 12th axiom without employ-

ing any new principle, was published by Franceschini, professor of mathematics in Bologna. This geometer demonstrates, that if B be a right angle, and A an acute one, and if CD, EF be perpendiculars to AD from the points C, E, AF must be greater than AD. This he does indirectly, by showing, that a straight line joining ED is not perpendicular to AB; and, secondly, that one drawn from E to any point between A and D, is likewise oblique to AB. He then infers, that as AE increases, AF must also increase, and that, since AE may be increased without limit, so AF may become greater than any given line, and may therefore exceed AB; wherefore, since the perpendiculars to AB, from points beyond B meet AE produced, the perpendicular BG must also meet it: and thus he conceives that he has proved the 12th axiom.

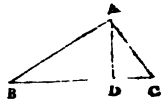


This reasoning, however, is fallacious, as magnitudes may continually increase, and yet never become equal to a given finite magnitude. Of this the 41st proposition of the third book of the Appendix affords an instance, as it is there shown, that the sum of all the lines CG, FL, GM, LN, &c. how many soever of them may be taken, will never become so great as the radius CF. Other instances are also furnished by the 40th and 42d propositions, by numberless infinite series', and by curves that have asymptotes, that is, lines which continually approach them, and yet never meet them. From the mere fact that a magnitude continually increases, we are not entitled to conclude, without considering the comparative magnitudes of the additions, that it will ever exceed a given amount: and therefore, in the present instance, we cannot legitimately infer, that, however AE is increased, the point F will fall beyond B.

3. In the Notes to Legendre's Eléments de Géométrie, a new method of investigating the theory of parallels is given, of which the following is an abstract. It is proved by superposition (or as in Euc. I. 26) that two triangles are in every respect equal, when a side and the two adjacent angles of the one are respectively equal to a side and the two adjacent angles of the other. If, therefore, p be a side of a triangle, and A and B the adjacent angles, the third angle C is determined if A, B, and p be given; as otherwise there might be different triangles which would have each a side equal to p, and the adjacent angles equal to A and B. Therefore the angle C must be a determinate function of A, B and p; or, as it may be expressed, $C=\phi(A, B, p)$, where ϕ is used to denote a function or combination of A, B, and p. Now p must be rejected from this equation; for, if not, we might have, conversely, the value of p expressed in terms of A, B and C, which is absurd, p and the angles A, B, C being heterogeneous, and such, that no combination whatever of the angles could give the value of p. Hence, therefore, we have simply $C=\phi(A, B)$; that is, C is determined by A and B, independently of the sides.

From this it follows, that if two angles of one triangle be equal to two angles of another, their third angles are also equal.

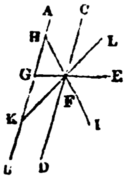
Let now ABC be a triangle having the angle BAC a right angle, and draw the perpendicular AD. Then, in the triangles ABC, DBA, the right angles BAC, BDA are equal, and the angle B is common; therefore BAD is equal to C. In like manner it would be shown, that CAD is equal to B: whence (I. ax. 2) the angle BAC is equal to the two angles B and C; but BAC is a right angle, and therefore the three angles of the right-angled triangle are together equal to two right angles.



The same property is easily shown to belong to any other triangle, as it may be divided by a perpendicular from the greatest angle to the opposite side, into two right-angled triangles, the angles of which will be equal to the angles of the assumed triangle, together with the two right angles made by the perpendicular—and this proves the 32d of the first book of Euclid.*

The author next proceeds to prove, on similar principles, that, in equiangular triangles, the sides which are similarly situated, are proportional, which is the 4th proposition of the 6th book of Euclid; and thence he derives at once a proof of the 12th axiom. This method, however ingenious, is much too abstract and difficult for an elementary treatise. The principle of homogeneity on which it depends, is in itself sufficiently obvious; but the mode in which it is here applied, would be intelligible to very few learners, when commencing the study of Geometry.

4. In Leslie's Elements of Geometry, a work which contains much curious and interesting matter, there is a singular failure in the attempt to prove that a straight line GE, falling on two parallels, AB, CD, makes the exterior angle EFC equal to the interior and remote angle EGA. The following is a correct abstract of his reasoning. Suppose a revolving straight line, extended in both directions, to pass always through F, and first, let it take the position IFH. Then, in the triangle HGF, the exterior angle EFH is greater (I. 16) than EGH. Again, let it take the position LFK. Then (I. 16) in the triangle FKG, the angle KFG, or its equal EFL, is less than EGK. When the incident line, therefore, meets AB above G, it makes an angle EFH *greater* than EGH, and when it meets AB below that point, it makes an angle EFL, which is *less* than the same angle. There is consequently a certain intermediate position, CD, in which the revolving line, not meeting AB on either side, and being therefore parallel to it, makes the exterior angle EFC equal to the interior EGA.



Now it is plain from the slightest consideration, that what is here demonstrated, is not what is intended, but simply that the straight line which makes the angle EFC equal to EGA, does not cut AB, but is parallel to it; which is the first part of Euclid's 28th proposition.

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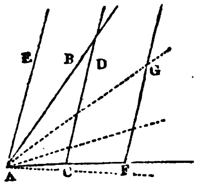
* It is evident, that the remaining part of the theory might be established without proportion, as in page 328.

5. A method of proving the 12th axiom has been proposed by M. Bertrand of Geneva, which is in substance the same as the following. Let AB, CD make with AC, the angles BAC, ACD, less than two right angles: AB, CD will meet, if produced. Draw AE, so that the angles EAC, ACD may be equal to two right angles: make also CF equal to AC, and the angle CFG equal to ACD. Then it would be shown by superposition, that the bands between AE, CD, and CD, FG are equal, if taken of the same length. Now, it is plain, that by the continued repetition of the angle EAB, as is shown by the dotted lines, the indefinite space contained between AE, AF, will be at length exhausted; but that same space can be exhausted by no number of repetitions of the indefinite band, or space contained between AE, CD, as would appear by constructing more such bands, on bases taken in AF produced, each equal to AC, or CF. Hence, therefore, the space contained between AE, AB, extended without limit, must be greater than the space contained between AE, CD, extended without limit; which, since AE is a common boundary of these spaces, could not take place, if AB lay wholly between AE and CD; and therefore AB must cut CD.



This method is very ingenious, and, on full consideration, will perhaps be regarded as satisfactory. The comparison, however, of the unlimited bands and angular spaces is scarcely conformable to the strict precision which we expect and admire in geometry. A resemblance will be readily traced between this method and that of Proclus, illustrated hereafter.

6. In Playfair's Elements of Geometry, the theory of parallels is founded on the following axiom, suggested by Ludlam in his Rudiments of Mathematics: "Two straight lines which intersect one another, cannot be both parallel to the same straight line. From this the author derives, indirectly, in a very simple manner, the proof of the first part of the 29th proposition: and the proofs of the second and third parts are then given as in Euclid. It seems, however, to have escaped the notice of these writers, that this axiom is virtually the same as Euclid's 12th axiom. To perceive this, it is only necessary to consider, that by the latter part of the 28th proposition, AGB is parallel to CHD, if the angles BGH, GHD be equal to two right angles; while it is assumed in the 12th axiom, that any other straight line drawn through G, will cut CHD, so that through G only one straight line can be drawn parallel to CHD.

7. The theory of parallels has been founded by Thomas Simpson, in the second edition of his Elements of Geometry, on the following axiom: "If two given points in a straight line are posited at unequal distances from another straight line, in the same plane, those two lines, being indefinitely produced on the side of the least

distance will meet one another." From this it is easy to prove, that a straight line falling on two parallels, makes the alternate angles equal. For by drawing lines through the points in which it intersects the parallels, perpendicular to one of them, and which therefore (I. 28) are parallel to one another, these perpendiculars are equal, as otherwise the given lines, by the axiom, would meet. For the same reason, the parts of the given lines between these perpendiculars are equal; and therefore (I. 8) the alternate angles are equal.

This method, if the axiom be admitted, is very simple and easy. When it is known, however, that the distance between two curves, or between a straight line and a curve, may continually decrease, and yet never vanish, it cannot be regarded as a perfect axiom, that two straight lines must meet, if their distances at different points be unequal.

8. Dr. Simpson, in the Notes to his edition of Euclid, admits that the 12th axiom is not self-evident; and gives a proof of it founded on the following axiom: "A straight line cannot first come nearer to another straight line, and then go farther from it, before it cuts it; and, in like manner a straight line cannot go farther from another straight line, and then come nearer to it; nor can a straight line keep the same distance from another straight line, and then come nearer to it, or go farther from it; for a straight line keeps always the same direction." This axiom is preceded by two definitions,* explaining the terms employed in it; and it is followed by five propositions, the last of which is the 12th axiom, and which therefore prepares the reader for the 29th proposition. Such a lengthened and prolix proof, as has been well remarked by Playfair respecting the method of Clavius, "leaves a strong suspicion, that the road pursued is by no means the shortest possible."

9. Clavius, after offering objections to the method of Proclus next to be explained, assumes as an axiom, that "a line, all the points of which are equally distant from a straight line in the same plane with it, is itself a straight line." Of this he endeavours to show the reasonableness; and though he does not demonstrate its truth, he concludes, that it will gain assent much more readily than the 12th axiom of Euclid. He then applies the assumed principle in demonstrating the 12th axiom, in doing which he employs a very tedious and operose process; and, on the whole, his method has little to recommend it.

10. Proclus, in his Commentaries, adopts the principle previously assumed by Aristotle, that *if two straight lines forming an angle, be extended infinitely from their point of intersection, their distance asunder will also increase, so as to exceed any given magnitude whatever.* On this assumption, he proves that a straight line

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[*] The latter of these concludes thus: "Two straight lines are said to keep the same distance from one another, when the distance of the points of one of them from the other, is always the same." This is liable to the same objections as the axiom of Clavius, mentioned in the next paragraph of the text.

which intersects one of two parallels, intersects the other also;[*] as the two lines which by hypothesis, intersect one another, may, by the assumed principle, be extended so far that their distance asunder will exceed the distance between the parallels, so that the line which cuts the one parallel must also cut the other. He then demonstrates the 12th axiom by showing that, of the two lines on which the third falls, one cuts a line drawn parallel to the other; and that consequently these lines must cut one another.

This method, were it developed with more care and rigour than have been employed by Proclus, would be preferable, on account of its conciseness, to several others; and the principle assumed, though not self-evident, will be admitted as readily as most others that have been proposed for the same purpose.

11. Many writers employ a principle similar to that of Clavius; but instead of calling it an axiom, they assume it as the definition of parallel lines. Thus, in Elrington's edition of Euclid, the method of Euclid is retained in the text; but in the Notes another method is given, as the foundation of which, parallel lines are defined to be those which are always equally distant, however far they are produced. This definition, which is also adopted by Boscovich, Wolfius, Pardies, Fernandez, Deschales, and Emerson, and by Thomas Simpson in the first edition of his Elements, is liable to objection on nearly the same grounds as the axiom of Clavius. If through two given points in a straight line, two equal perpendiculars be drawn on the same side, one straight line (I. post. 1) and (I. def. 3, cor.) only one can be drawn through their other extremities: and a definition in which it would be assumed, that this line is parallel to the proposed line, would be perfectly legitimate. Without proof, however, we are not entitled to infer, that other perpendiculars from the first of these lines to the second are each equal to one of the first two perpendiculars, or that a perpendicular to one of the lines, is also perpendicular to the other, or finally, that the two lines will never meet, however far they may be extended. These propositions, though all true, require proof; and if we employ them without it, we should at least be aware, that we do what is inconsistent with strict reasoning.

12. A method different from all the foregoing is given in Cresswell's Treatise of Geometry, published in 1819. He assumes the principle, that *through any point within an angle a straight line may be supposed to pass, which shall cut the two straight lines that contain the angle.*[†] Admitting this axiom, he demonstrates, that if ABD be a right angle, and BAC an acute one, BD and AC will meet, if produced through C and D. To prove this, he makes

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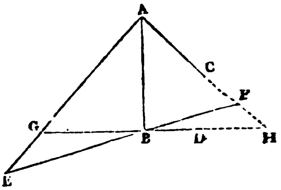
[*] This principle is assumed without proof in Bonnycastle's Elements, as the basis of the theory of parallels; and instead of being called an axiom, it is improperly classed among the postulates.

[†] This principle, which is tacitly assumed by Euclid, in the 20th proposition of the eleventh book, is easily derived from Euc. I. ax. 12, and I. 31, when the theory of parallel lines is established by other means.



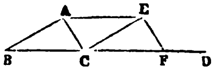
the angle BAE equal to BAC. Then, by the axiom, some line EBF may be drawn cutting AC, AE. If this coincide with BD, the proposition is plain; as it is also if BF fall below BD. But if BF fall within the angle ABD, DB produced to G will fall within the angle ABE, and will therefore cut AE, since BE cuts AE. The proof is then completed by applying the angle BAE to BAC, so that AE may fall on AC; and BG will fall on BD; whence the truth of the proposition is evident.

From this proposition the theory of parallel lines is easily derived; and the assumption, or axiom, will perhaps be as readily admitted, as the others that have been assumed for the same purpose.

The following method of establishing the theory of parallel lines, which occurred to me several years ago, I shall venture to insert here. I am aware that it is liable to some of the objections that have been made against other methods. At the same time, the axiom on which it is founded, will be more readily admitted than some others which have been proposed for the same purpose; and it gives short and simple proofs of the 32d proposition, and the 12th axiom. The 29th proposition may then be demonstrated in the manner given by Euclid.

Axiom. — If a triangle be moved along a plane, so that its base may always be on the same straight line, its vertex describes a straight line equal to that which is described by either extremity of the base.[*]

Book I. Prop. XXXII.—If a side of any triangle be produced, the exterior angle is equal to the two interior and remote angles; and the three interior angles of every triangle are together equal to two right angles.



Let the side BC of the triangle ABC be produced to D: the exterior angle ACD is equal to the two interior and remote angles CAB, ABC; and the three interior angles of the triangle are together equal to two right angles.

Let the triangle ABC be moved along the plane in which it lies, so that the base BC may always be on the straight line BD, till the point B shall fall on C, and the triangle ABC take the position ECF, the point A describing, by the axiom, the straight line AE equal to the base BC, described by the point B. Then in the two triangles ABC, CEA, the side EC is equal to AB, the side AC is common, and the base AE is equal to the base BC: wherefore (I. 8) the angle ACE is equal to the angle BAC. To these angles add ECF and ABC, and the whole exterior angle ACD is equal (I. ax. 2) to the two interior and remote angles CAB, ABC. To these, again, add the angle ACB, and the three interior angles are equal (I. ax. 2) to the two angles ACD, ACB; that is, (I. 13) to two right angles. Wherefore, if a side, &c.

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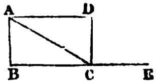
[*] This axiom might be generalized; since it is evident, that the same will hold respecting any figure whatever, having a straight line for one of its sides, and respecting any point whatever in that figure. As given above, however, it is sufficient for the intended purpose.

Cor. 1. Hence the interior angles of a quadrilateral are together equal to four right angles, being equal to the angles of the two triangles into which it is divided by either diagonal: and therefore if three angles of a quadrilateral be together equal to three right angles, the remaining angle is a right angle.

Cor. 2. If a quadrilateral ABCD have all its angles right angles, its opposite sides are equal. Join AC, and produce BC to E. Then (I. 13) DCE is a right angle; and (I. 32) the angle ACE is equal to the two angles ABC, BAC. From these equals take away the right angles DCE, ABC, and there remains the angle ACD equal (I. ax. 3) to the angle BAC. Hence, the angles D and B being equal, and the side AC common to the two triangles ADC, CBA, the side AD is equal (I. 26) to BC, and DC to AB.



DEMONSTRATION OF AXIOM 12.*—If a straight line meeting two other straight lines in the same plane, make the two interior angles on the same side of it together less than two right angles, these straight lines will at length meet on that side, if they be continually produced.

Let the straight line AB falling on the two straight lines AC, BD, make the angles CAB, ABD, together less than two right angles: AC and BD, if continually produced, will meet towards C, D.

Of the angles CAB, ABD, one at least must be acute; let ABD be acute, and draw (I. 12) AE perpendicular to BD. This perpendicular will fall (I. 16. cor. 2) on the side of B towards D. Then the angle AED is equal (I. 32) to ABD, BAE: to each of them add EAC, and the angles AED, EAC are together equal to CAB, ABD, and are therefore less than two right angles; but (const.) AED is a right angle, and therefore EAC is less than a right angle.

Through A draw AF perpendicular to AE: take any point G in AC, and make GH equal to AG; draw (I. 12) GK, HL per-

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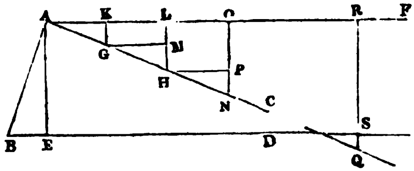
[*] This demonstration, I believe to be new. A more concise proof is given by Playfair, who, in his note on the 29th proposition, derives the 32d in a manner quite different from that which has been here employed. His proof of the axiom, however, presupposes the knowledge of the first lemma of the twelfth book, which is not required in the demonstration here given.

pendicular to AF, and GM to HL. Then, since (const.) three angles of the quadrilateral KM, are right angles, KGM is also a right angle, by the first of the preceding corollaries. Now (I. 32) the exterior angle KGH is equal to the two interior and remote angles GKA, KAG. From these equals take the right angles



KGM, AKG, and (I. ax. 3) there remains the angle MGH equal to KAG: also the right angles GMH, AKG are equal, and (const.) the side GH is equal to AG. Hence (I. 26) in the triangles GMH, AKG, the sides HM, GK are equal. But, by the second of the preceding corollaries, ML is also equal to GK; wherefore HM, ML are equal, and HL is double of KG.

In like manner, if HN be taken equal to AG or GH, and if NO be drawn perpendicular to AF, and HP to NO, it might be proved that NP is equal to HM or GK, and NO treble of KG; and by continuing this process, a point may be found in the straight line AC, such that the perpendicular drawn from it to AF may be equal to KG taken any number of times whatever; and may therefore exceed any given line.*

Let Q be a point found in this manner, in the continuation of AC, such that if QSR be drawn perpendicular to AF, QR will be greater than the perpendicular AE. Then, by the first of the foregoing corollaries, in the quadrilateral ER, the angle ESR is a right angle; and, by the second corollary, SR is equal to EA: but QR is greater than EA, and therefore it is also greater than SR. Hence the points A and Q are on opposite sides of BD; and therefore AC and BD, being continually produced, intersect each other.

PROP. XLIV. AND XLV.

These two propositions, as well as several others, may be advantageously postponed till the student has made himself acquainted with some of the principal propositions of the sixth book: and this may be done without violating any principle of good reasoning, as the propositions in the sixth book depend in no degree on these. The conciseness and ease of the following solutions will show the advantage of this change of arrangement.

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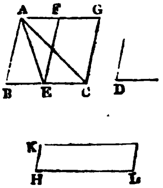
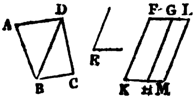
* It is here assumed as an axiom, that the less of two given magnitudes may be multiplied so as at length to exceed the greater. The same is done by Euclid, in the proof of the eighth proposition of the fifth book; and also in that of the first proposition of the tenth book, or, in this edition, in the first lemma in the twelfth book.

PROP. XLIV.—Let ABC be a given triangle, D a given angle, and HK a given straight line: it is required to apply to HK a parallelogram equal to ABC, and having an angle equal to D.



Bisect (I. 10) BC in E, and (I. 23) make the angles CEF, KHL each equal to D; draw (I. 31) AG parallel to BC, and CG parallel to EF. Then (I. 42) the parallelogram EG is equal to ABC. To HK, EC, EF find (VI. 12) a fourth proportional, and make HL equal to it; complete the parallelogram KL: it is the one required. For (const.) HK : EC : : EF : HL; that is, in the two parallelograms FC, KL, the sides about their equal angles FEC, KHL, are reciprocally proportional; wherefore (VI. 14) KL is equal to FC, and therefore (I. ax. 1) to ABC. Hence KL is applied to HK, is equal to ABC, and has an angle equal to D: it is therefore the parallelogram required.

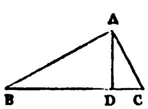
PROP. XLV.—Let ABCD be a given rectilineal figure, and E a given angle: it is required to describe a parallelogram equal to ABCD, and having an angle equal to E.



Join DB, and describe (I. 42) the parallelogram FH equal to the triangle ABD, and having the angle K equal to E. Produce FG, KH through G and H. Then, as in the preceding proposition, bisect BC, and through the point of bisection draw a straight line making an angle with one of the halves of BC equal to K, and terminated by a parallel to BC through D. To HG, to the half of BC, and to the line drawn through the point of bisection of BC, find (VI. 12) a fourth proportional; make HM equal to this, and draw ML parallel to KF or HG. Then it would be shown, as in the preceding proposition, that GM is equal to BCD, and consequently (I. ax. 2) FM is equal to AC; and, having the angle K equal to E, it is the parallelogram required.

PROP. XLVII.—This celebrated proposition may also be demonstrated with great ease by means of proportion, in the following manner.

Let ABC be a triangle, right-angled at A; the square of the hypotenuse BC, is equal to the squares of the legs AB, AC.



Draw AD perpendicular to BC. Then (VI. 8, cor.) BC : BA : : BA : BD, and BC : CA : : CA : CD. Hence (VI. 17) the rectangle BC.BD is equal to the square of AB, and the rectangle BC.CD to the square of AC. Hence (I. ax. 2) BC.BD + BC.CD, or (II. 2) $BC^2 = AB^2 + AC^2$.

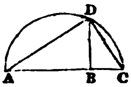
Other proofs will be found in the first book of the Appendix, propositions 9, 10, and 11.

ELEMENTS, BOOK II.

Some propositions in this book may be demonstrated with much ease by means of propositions in the sixth book. The following proofs of the fourth, fifth, and fourteenth propositions afford instances.

Prop. IV.—Let the straight line AC be divided into any two parts in B: then $AC^2 = AB^2 + BC^2 + 2AB.BC$.

Draw BD perpendicular to AC, and meeting a semicircle described on AC as diameter in D; and join DA, DC. Then (III. 31) ADC is a right angle. Also (I. 47) $AD^2 = AB^2 + BD^2$, and $CD^2 = BC^2 + BD^2$. Add these equals; then $AD^2 + CD^2$, or (I. 47) $AC^2 = AB^2 + BC^2 + 2BD^2$. But (VI. 8, cor.) AB : BD :: BD : BC; whence (VI. 17) $AB.BC = BD^2$: and therefore by substitution, $AC^2 = AB^2 + BC^2 + 2AB.BC$.



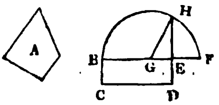
Prop. V.—Let the straight line BF (see the next figure) be divided equally in G, and unequally in E: then $BE.EF + GE^2 = GF^2$.

For if BH, FH were joined, it would appear, as in the preceding proposition, that the rectangle BE.EF is equal to the square of EH. To each add the square of EG: then $BE.EF + EG^2 = EH^2 + EG^2 = GH^2 + GF^2$.

The sixth proposition might be demonstrated on similar principles. The second proof in the text, however, is as easy as any that can be given.

Prop. XIV.—Let it be required to describe a square equal to the given rectilineal figure A.

Describe (I. 45) the rectangle BD equal to A. Then, if the sides BE, ED be equal, BD is the required figure. But, if they be unequal, find (VI. 13) EH a mean proportional between them. Then (VI. 17) the square of EH is equal to the rectangle BE.ED; that is, to the given figure A; therefore EH is the side of the required square.



ELEMENTS, BOOK III.

Prop. XXXV. and XXXVI.

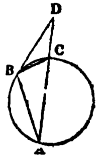
The 35th and 36th propositions of this book may each be demonstrated in a single case by means of proportion in the following manner.

PROP. XXXV.—Let the chords AE, BC (see the figure, page 235) in the circle ABEC, cut one another in G: the rectangles AG.GE, BG.GC are equal.

Join AC, BE. Then (I. 15) the angles AGC, EGB are equal, and III. 21) C is equal to E : wherefore (VI. 4, cor.) AG : GC :: BG : GE, and therefore (VI. 16) AG.GE = CG.GB.

PROP. XXXVI.—From D, a point without the circle ABC, let the straight line DB be drawn touching the circle, and DCA cutting it: the rectangle AD.DC is equal to the square of DB.

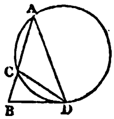
Join BA, BC. Then, in the triangles DBA, DCB, the angle D is common, and (III. 32) the angle A is equal to CBD; wherefore (VI. 4. cor.) AD : DB :: DB : DC; and therefore (VI. 17) $AD.DC = DB^2$.*



ELEMENTS, BOOK IV.

PROP. X.—This may be demonstrated in the following easy manner by means of proportion, the circle being unnecessary.

By construction, $AB.BC = AC^2 = BD^2$; whence (VI. 17) AB : BD :: BD : BC; and therefore (VI. 6) the angle BCD is equal to ADB, or its equal B. Hence (I. 6) CD is equal to BD, or to its equal AC, and (I. 5) the angle A is equal to ADC. Now BCD, which is equal to B, being equal (I. 32) to both these taken together, is double of one of them: therefore B is double of A.



ELEMENTS, BOOK V.

Several minute changes, and, it is hoped, improvements, have been made in this book, especially in the definitions. These it is unnecessary to specify; and I shall merely subjoin some remarks, which will tend to illustrate in a familiar manner, the nature of proportion generally, and in particular the fifth and sixth definitions of this book. Our common idea of proportion, and that which is presented in these definitions, are apparently very different; and it will be a principal object in the following observations to show their agreement.

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* The study of Euclid would be considerably facilitated to the beginner, if, in accordance with some of the preceding Notes, he should omit in the first reading, the 44th and 45th propositions of the first book, and all in the second book after the first three: then, omitting in like manner, the 35th, 36th, and 37th of the third book, and perhaps all the fourth book, he should proceed to the fifth and sixth books. Then, having advanced in the latter to the 17th proposition, he may turn back and read what he has omitted, availing himself of the facilities that will be afforded, in several instances, by means of the sixth book. He might also postpone the 47th of the first book: but Euclid's proof of it is so elegant and easy, as to render this unnecessary; and it is desirable, that he should be early acquainted with so important a proposition.

In estimating the comparative magnitudes of numbers or other quantities, we generally employ a species of division. Thus, we say, that one country has three times or four times the extent of another; and that the population of one city exceeds that of another in a fourfold degree; thus considering how often one of the magnitudes contains the other, and obtaining by that means the idea of ratio. If by such an examination we find that one country contains three times as many square miles as another, and that the population of the former is treble of that of the latter, we see that the ratios of the extents and populations are equal, and we say that the populations of the countries and their extents are proportional. The method, therefore, which naturally presents itself for estimating the ratio of one quantity to another, is to divide the former by the latter. Thus, the ratio of a line of 10 inches to one of 2 inches is 5, or is that of 5 to 1; and since, on the same principle, 15 has to 3 the same ratio, we say that 10, 2, 15 and 3 are proportionals. According, therefore, to this first and simplest notion of proportion, it appears that *four magnitudes are proportional, when the quotients obtained by dividing the first by the second, and the third by the fourth, are equal.* These quotients, in the great majority of actual instances, are fractional: still, however, if they be equal, we regard the magnitudes as proportional. Thus, 15, 6, 20, and 8 are proportional, because 15 is equal to twice 6 and the half of 6, and 20 to twice 8 and the half of 8; or, which is the same, the quotients are each $2\frac{1}{2}$. In like manner, 6 is to 8, as 9 to 12, each of the quotients being in this instance the fraction $\frac{3}{4}$.

Now, by the nature of division, if the second and fourth terms be multiplied by the quotient, found as above, the products will be the first and third terms: and it evidently follows, that if the first and third terms be multiplied by any number whatever, and the second and fourth by the product of that number and the same quotient, the products of the first and second terms will be equal, as will also those of the third and fourth. Thus, resuming the numbers 15, 6, 20, and 8, if we multiply 6 and 8 by $2\frac{1}{2}$, the products are 15 and 20; that is, $15 = 2\frac{1}{2} \times 6$, and $20 = 2\frac{1}{2} \times 8$. If we now multiply these equals by any number whatever, the products will be equal. Let us, therefore, multiply them by the denominator 2, and we obtain $2.15 = 5.6$, and $2.20 = 5.8$. From this it appears, that when the antecedents are multiplied by 2, and the consequents by 5, the multiples of the first and second are equal, as are also those of the third and fourth: a result which agrees with *one of the conditions* of Euclid's fifth definition: and it is plain, that if, instead of the multiples 2 and 5, we should employ their doubles, trebles, or any other numbers in the same ratio, we should still arrive at a similar conclusion. It is also plain, that if, retaining 5 as one of the multipliers, we were to use along with it any number greater than 2, the multiples of the first and third would exceed those of the second and fourth: while if 2 were retained, and with it any multiplier greater than 5 were used, the multiples of the first and third would then be less than those of the other terms: and thus we have the agreement with the *two remaining conditions* of Euclid's definition established in the instance which we have been considering.

In this example, 3 is a common measure of the first and second terms, and 4 of the third and fourth; and the numbers 5 and 2 are obtained by dividing the terms of the two ratios respectively by these: and in the same manner, in every case in which the terms of the ratios are commensurable, we may find numbers by which, if they be multiplied, the products will be equal. If, however, these terms be incommensurable, that is, if they have no common measure, no numbers can be found which will give multiples exactly equal, as the quotient of two incommensurable quantities cannot be accurately expressed by any number either whole or fractional. We may find numbers, however, which will render the multiples as nearly equal as we please. Thus, suppose the terms of the ratio to be $\sqrt{2}$ and 1, the former being the diagonal of a square of which the latter is the side. Then, since, by extracting the square root of 2 numerically, we find 1·414... for the approximate value of the first term, the quotient obtained by dividing the first by the second is greater than $\frac{14}{10}$ and less than $\frac{15}{10}$: and if we multiply the first term by 10, and the second by 14, the first of the two results is greater than the second; while, if 10 and 15 be employed as multipliers, the first product is less than the second. If, again, 100 and 141 were employed as multipliers, and also 100 and 142, the multiples would still have the same relation, but would approach more nearly to the ratio of equality: and thus, by taking other multipliers, we might approximate to exact equality, as nearly as we please. Now, it is evident, that every thing would be precisely similar with respect to $2\sqrt{2}$ and 2, $3\sqrt{2}$ and 3, and so on: so that $\sqrt{2} : 1 :: 2\sqrt{2} : 2 :: 3\sqrt{2} : 3$, &c. It thus appears that not only in commensurable magnitudes, but also in incommensurable ones, if the quotients obtained by dividing the antecedents by the consequents be equal, the magnitudes are proportional, according to the criterion of Euclid: for if multiples be taken, as in the 5th definition, those of the antecedents will be either both greater, or both less, than those of the consequents; and if the multiples of the first and second terms be nearly in the ratio of equality, so will those of the third and fourth.

The illustration here given is the same in substance as proposition I. page 117. Elrington, Leslie, and others, instead of Euclid's definition, have adopted one to the following effect: *The first of four magnitudes is said to have to the second the same ratio which the third has to the fourth, when any submultiple whatever of the first is contained in the second, as often as a like submultiple of the third is contained in the fourth.* This definition readily arises from the supposition, that when four magnitudes are proportional, the first term is contained in the second as often as the third is contained in the fourth, which, with the exception of the order in which the terms are taken, is the same as the popular idea of proportion already considered. If, in applying this principle, we either find that the first and third terms are greater than the others, or are not contained in them without remainder, we use, instead of the first and third, submultiples of them, and we regard the magnitudes as proportionals, when the first, or any submultiple of it, is contained in the second, as often as the third, or a like submultiple of it is contained in the fourth. This method is very simple in its application, and, with proper management, it serves to establish the theory of proportion on correct principles.

With regard to the Supplement to the fifth book, it may be remarked, that while the properties established in it are primarily those of numbers, they belong equally to lines, surfaces, and other magnitudes; as all magnitudes whatever may be conceived to be divided into parts, each equal to a common unit, and may thus be expressed by means of numbers.

ELEMENTS, BOOK VI.

DEF. V.

Instead of this definition, the following is sometimes adopted: "Three or four straight lines are said to be in harmonical proportion, when the first is to the last as the difference of the first two to the difference of the last two." Thus, in the figure for proposition A, page 130, the three lines DB, DG, DC are in harmonical proportion, if DB : DC :: BG (=DB—DG) : GC (=DG—DC); which agrees exactly with the definition given in the text, if the straight line DB be regarded as divided into the three parts DC, CG, GB.

This relation of lines receives its denomination from the fact, that three musical strings of the same thickness and tension, and proportional to 1, $\frac{2}{3}$, and $\frac{1}{2}$, produce the sounds of a certain note, its fifth, and its octave; and these lengths are such that the first is to the third, as the difference of the first and second to the difference of the second and third. By the insertion of other harmonical means between these, the lengths of the strings, giving the remaining notes of the octave, will be obtained, the lengths for the eight notes being proportional to the numbers 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$: of which $\frac{4}{5}$ is an harmonical mean between 1 and $\frac{2}{3}$; $\frac{8}{9}$ between 1 and $\frac{4}{5}$; and $\frac{3}{4}$ the second of two harmonical means between 1 and $\frac{2}{3}$: while $\frac{3}{5}$ is the first of two such means between $\frac{2}{3}$ and $\frac{1}{2}$; and $\frac{8}{15}$ is the harmonical mean between $\frac{1}{2}$ and the harmonical mean between it and $\frac{2}{3}$.*

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* The reciprocals of three equidifferent numbers, or, as they are commonly called, numbers in *arithmetical progression*, are in harmonical proportion. Thus, 1, $1\frac{1}{2}$, and 2, have equal differences; and their reciprocals, that is, the quotients obtained,

PROP. XXVIII. AND XXIX.—These propositions may be omitted by the student, as they are of scarcely any use, except the cases of them which are the 35th and 36th propositions of the Appendix, Book I. From the latter propositions, too, it is easy to derive solutions of the former. Thus, to construct the 28th, make (I. 45) a rectangle equal to the given figure K: then to the base EC, to the altitude of CD, and to one side of the rectangle equal to K, find a fourth proportional, and complete the rectangle contained by this fourth proportional and the other side of the rectangle equal to K: lastly, divide (APP. I. 35) AB in X, so that the rectangle AX.XB may be equal to the latter rectangle, and the thing is done. The proof is easy, and may serve as an exercise to the student. The 29th will be constructed in exactly the same manner, except that the 36th, and not the 35th, of the Appendix, Book I. is to be employed.

PROP. XXX.—From this proposition, and from the scholium to the eleventh of the second book, it is plain that the segments of a line cut in extreme and mean ratio, are incommensurable to one another. For the greater segment is to the less, as the less to their difference: the less to the difference, as the difference to the difference between it and the less, and so on continually; no remainder being ever found, which is contained exactly in the preceding one. See page 263.

It thus appears, that the ratio of the parts of a line divided in this manner, cannot be exactly expressed by any two numbers. It may be approximated, however, as nearly as we please, by means of the following series of numbers, the successive terms of which are each equal to the sum of the two immediately preceding it, as is the case in a reversed order in the series of lines obtained in the scholium to the eleventh of the second book:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, &c.

Thus the parts are nearly as 3 to 5, more nearly as 5 to 8, more nearly still, as 8 to 13, and so on.

PROP. D.—If R be put to denote the radius of the circumscribed circle, p the perpendicular, S the area, and a, b, c the sides opposite to the angles A, B, C, we have from this proposition $bc = 2Rp$. Multiply by a: then $abc = 2Rpa$, or $abc = 4RS$, because $pa = 2S$. Hence, if the three sides of a triangle be given in numbers, we may compute the radius of the circumscribed circle, by dividing the continued product of the three sides by four times the area, computed (APP. IV. 29) from the sides.

Hence also it appears (APP. V. 4) that a parallelepiped, having

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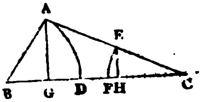
by dividing unity by them, are 1, $\frac{2}{3}$, and $\frac{1}{2}$, the harmonicals mentioned above. Again, if there be three harmonicals, and if the first and second be both multiplied, or both divided, by any number, and the second and third be both multiplied, or both divided, by any other number, the four numbers thus obtained are in harmonical proportion. These properties are most easily proved by means of algebra.

for its length, breadth, and altitude, the three sides of a triangle, is equal to a prism having the triangle for its base, and twice the diameter of the circumscribed triangle for its altitude.

TRIGONOMETRY.

PROP. II.—Some may perhaps prefer the following proof of this proposition to the one given in the text.

From B as centre with BA as radius, describe the arc AD; and from C as centre, with an equal radius, describe the arc EF. Draw AG, EH perpendicular to BC; these (APP. III. def. *5*) are respectively the sines of B and C to equal radii. Then the triangles AGC, EHC are equiangular, the angle at C being common, and the angles at G and H right angles. Hence (VI. 4) CA : AG :: CE, or (const.) AB : EH; and, alternately, CA : AB :: AG : EH; that is, $b : c :: \sin B : \sin C$.



The demonstration is simplified by taking, as here, one of the sides, AB or AC, as radius. This, however, is not essential, as arcs may be described from B and C as centres, with equal radii of *any* magnitude, and their sines, and a perpendicular from A to BC being drawn, the proof will be readily obtained.

CASE II. Page 170.—The method which has been given for solving this case is, I believe, the best of which it is susceptible. The method of finding the third side, after half the difference of the angles has been computed, was first pointed out by Professor Wallace of Edinburgh, in the tenth volume of the Edinburgh Transactions; and, in conjunction with the preceding analogy, requires the tables to be opened in only five places for the entire solution and verification. The solution depends on the third and fourth propositions preceding it, of the latter of which I have given a very short and easy proof, which I believe to be new.

If the remaining side, instead of being found by the method here adopted, by computed by means of the first case, by two different analogies for the sake of a verification, the tables, as has been remarked by Professor Wallace, must be opened in no fewer than ten different places; and thus we see how much preferable the one method is to the other.

CASE III. Page 170.—The last method of solving this case was first published, so far as I know, in 1830, in the second edition of my "Elements of Plane and Spherical Trigonometry." It is preferable in a considerable degree to any of the other methods, when all the angles are to be computed, with a view to obtain a verification; as the operations are more simple and easy, and require only four logarithms, while seven are required in any of the other methods. In the first method also, a practical difficulty is often experienced in finding the segments of the base with sufficient precision; and a slight inaccuracy in assigning them, may produce a very perceptible error in the values obtained for the angles; — an inconvenience which does not belong to the new method, or to the second or third. An example of the mode of employing the new method is given in the note in page 171.

ELEMENTS, BOOK XI.

Def. VIII. — Instead of this definition the following is given in Dr. Simson's edition: "Similar solid figures are such as have all their solid angles equal, each to each, and which are contained by the same number of similar planes." This, instead of being a definition of similar solid figures in general, is only a definition of those which are contained by planes, and it should be restricted accordingly.

Prop. IV.—A simple and easy demonstration of this proposition has been substituted for Euclid's. It is the same in substance as that which was given by the late Professor Wallace of Edinburgh, in his editions of Playfair's Geometry.

Prop. VI. — This may also be demonstrated in the following manner. Draw DE of any length perpendicular to BD, and join AE, BE, AD. Then $AE^2 = AB^2 + BE^2 = AB^2 + BD^2 + DE^2 = AD^2 + DE^2$; wherefore (I. 48) ADE is a right angle. The rest of the proof is the same as in the text.

Prop. XXIII. — By means of a lemma, a new demonstration of this proposition is given, which is much shorter and easier than that of Euclid.

Prop. XXV.—In this proposition and many others, Dr. Simson, rendering the Greek literally, uses the expression "solid parallelepiped." In this edition, the word "solid" is omitted as superfluous, in this and similar instances.

In connexion with this it may be mentioned, that in the 16th proposition of the first book, and many others I have used the phrase, *remote angle*, instead of *opposite angle*, confining the latter expression to the case in which an angle is mentioned in connexion with a straight line which subtends it, and to the one in which two remote angles of a quadrilateral are spoken of. I have also, after Dr. Simson, avoided the use of the expression *right* line; adhering in both this and the preceding instance, to the principle, that in the language of science, as well as in other cases, words should be used as nearly as possible, in their ordinary and obvious meaning.

Prop. B.—When, according to the hypothesis of this proposition, the solid angles at A and B are contained by plane angles which are not only equal, each to each, but which are *similarly situated*, so that if the planes CAD, FBG be made to coincide, the edges AE, BH lie on the same side of the coincident planes, the solid angles wholly coincide, and are therefore equal. But if the plane angles forming the two solid angles be arranged in a *reversed order*, so that when CAD, FBG are made to coincide, the edges AE, BH will be on opposite sides of the coincident planes, the solid angles will not coincide. Still, however, by proposition A, the inclinations of the planes forming the solid angles will be equal, and we can have no hesitation in admitting, that the solid angles are themselves equal. Such solid angles as these, that is, such as are equal, and yet from lying in opposite directions, cannot coincide, have been denominated by Legendre, *angles equal by symmetry*, or simply *symmetrical angles.*

On the same principle, there may be *symmetrical solids;* such, for instance, as two pyramids similarly described on opposite sides of the same base, and which, though obviously equal, cannot be made to coincide by superposition.

Between plane figures and solids, there is a striking difference with regard to the present subject of consideration, as will appear by a simple example. If a perpendicular be drawn from the vertex of an isosceles plane triangle to the base, one of the parts into which the proposed triangle is thus divided, may be inverted and made to coincide exactly with the other.[*] If, however, through the point in which the perpendicular cuts the base, a straight line, unequal in magnitude to half the base, be drawn perpendicular to the plane of the triangle, and through its extremity straight lines be drawn to the several angles, a pyramid will be formed which will evidently be bisected by the plane passing through the perpendicular of the triangle and the perpendicular to its plane; and yet the halves of the pyramid cannot be made to coincide.

The principle on which we infer the equality of such solids, is, as has been remarked by Playfair, that the conditions which determine the magnitude of one of them, are the same as those which determine the magnitude of the other.

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[*] The case is different with regard to an isosceles spherical triangle; as, while the perpendicular will evidently divide it, as well as the perpendicular divides the plane triangle, into two triangles, having equal sides, equal angles, and equal areas, those triangles cannot be made to coincide, unless the sides of the isosceles triangle be quadrants.

It may be worth remarking, that, if without inverting either of the parts into which the perpendicular divides an isosceles plane triangle, the vertical angles be made to coincide, the parts of the triangles which do not coincide, will be two triangles having their sides and angles equal, each to each: that if the vertical angles of these be applied to one another in a similar manner, the parts which do not coincide will also be triangles, having equal sides and angles, and so on; and if this process be continued, two triangles will at length remain, which will exactly coincide, if the angles of the proposed isosceles triangle be commensurable with a right angle: while if they be not, the process may be continued, till the triangles, which do not coincide, will be less than any assigned space. The proof of this is easy: and it will be readily seen that there may be an analogous subdivision in case of an isosceles spherical triangle.

It may also be remarked, that if about any two such plane triangles, circles be described, and straight lines be drawn from their centres to the several angles, these lines and the sides of the triangles will form triangles, which will exactly coincide with each other: and the same will hold regarding spherical triangles, if arcs of great circles be drawn from the several angles to the nearer pole of the circumscribed circle; so that though such triangles cannot coincide as wholes, yet they may be divided into parts which will coincide; and the like relation will evidently hold regarding trihedral angles, which are equal by symmetry.

It is easy to see from these considerations, that, as was first shown by Dr. Simson, solids may be unequal, though contained by the same number of equal and similar planes. Thus, as a simple example, suppose two symmetrical pyramids described on opposite sides of the same base: then, if on the same base another pyramid be described, lying wholly without the one of these, which is on the same side of the base with it, the whole solid thus formed, and the solid which remains when both the symmetrical pyramids are taken away, are evidently contained by the same number of equal and similar planes; while at the same time one of them is only a part of the other. It is plain also, that while the solid angles of these two unequal solids, are respectively contained by the same number of equal plane angles, some of these solid angles are unequal. No solid angles, indeed, except trihedral ones, are determined without more data than the plane angles which contain them.

The nature and relations of solid angles are well illustrated by means of spherical trigonometry; since, if a sphere be described from the vertex of a solid angle as centre, the solid angle will be measured by the portion of the spherical surface bounded by the planes forming the angle. On this subject the reader may consult Gregory's Trigonometry, Chap. VI. Section 6.

PROP. XXXV.—The demonstration of this proposition is greatly shortened and simplified, chiefly by means of a corollary which has been annexed to proposition B.

APPENDIX, BOOK I.

PROP. VI.—This proposition has been inserted on account of its utility in the solution of many problems regarding triangles.

PROP. XX.—Of this proposition, which is the tenth of the thirteenth book of Euclid, I have given a demonstration which I believe to be new, and which is much easier than that of Euclid.

PROP. XXVI.—This curious proposition I have not met with, except in the "Dublin Problems," published in 1823, where it is inserted without demonstration. The following is an outline of a very easy and neat proof of it by Mr. Adam D. Glasgow of Belfast, a former student of mine, of great taste and talent for mathematical pursuits:—

If circles be described about the triangles ABD, ACE, and if straight lines be drawn from A, B, C to their intersection, which suppose to be marked with L, the angles ALB, ALC will be each two thirds of two right angles, being (III. 22) supplements of D and E: and hence, since there remain two thirds of two right angles for the angle BLC, it follows (APP. I. 4) that the circle described about BCF will also pass through L. It is also plain (see the second note to the tenth proposition of the first book) that GK cuts AL perpendicularly, suppose in M; and that GH cuts BL perpendicularly, suppose in N. Then, since the angles at M and N are right angles, a circle (APP. I. 4) may be described about GMLN: and therefore, since MLN is two thirds of two right angles, MGN (III. 22) is one third of two right angles. In the same manner it may be proved that GHK, HKG are each a third of two right angles: and hence (I. 6, cor.) GHK is an equilateral triangle.

PROP. XXXV. AND XXXVI.—These two propositions might be otherwise enunciated thus: "Given the sum of two straight lines, and the mean proportional between them; to determine them:" and "Given the difference of two straight lines, and the mean proportional between them; to determine them."

PROP. XLI. AND XLII.—In the year 1774, the interesting problem mentioned in the corollary to proposition XLI. was published, without solution, by the Rev. Mr. Lawson. The method of solving it was soon after discovered by Dr. Charles Hutton, and was acknowledged by Mr. Lawson to be on the same principle as his own. See Hutton's Tracts, Vol. I. page 254, and Vol. III. page 378; where also information will be found regarding proposition XLII.; the best mode of solving which seems also to be due to Dr. Hutton.

PROP. XLIII.—This proposition, which is curious, as affording the first instance of the quadrature of a curvilineal space, is generally supposed to have been discovered by Hippocrates of Chios, an ingenious mathematician, who flourished about 450 years before Christ. For additional information respecting lunes, see Hutton's translation of Ozanam and Montucla's Recreations, Vol. I.

APPENDIX, BOOK II.

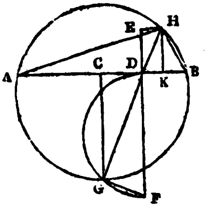
The second book of the Appendix contains, in a small compass, much interesting matter. It commences with solutions of the "tangencies;" and it then introduces the student successively to the consideration of loci, of porisms, and of isoperimetrical figures.

APPENDIX, BOOK III.

This book contains a miscellaneous collection of propositions, intended to apply and extend the principles established in the preceding part of the work. To the solutions of several of the problems the method of computation is subjoined; and in this way the student will obtain an interesting application of plane trigonometry.

Prop. XVI.—This problem may be solved in the following manner, without employing proposition G of the sixth book:

Let AB be the given base, and let it be bisected in C, and divided in D so that AD may be to DB in the ratio of the sides. Draw DE at right angles to AB, and equal to the perpendicular, found by means of the corollary to the 45th proposition of the first book; and produce it through D, making DF equal to a fourth proportional to ED, AD, DB. On DF as diameter describe a semicircle; and draw CG parallel to EF, and meeting its circumference in G. Draw EH parallel to AB; and draw the straight line GDH, meeting it in H. Lastly, join AH, BH; and AHB is the triangle required.



Join GF. Then, in the equiangular triangles DEH, DGF, as HD : DE :: FD : DG; and (const.) DE : AD :: DB : DF. Hence (VI. 16) HD.DG = DE.DF, and DE.DF = AD.DB; and, consequently, HD.DG = AD.DB, and (App. I. 5) a circle will pass through A, G, B, H, the centre of which (III. 1, cor.) will be in CG, since CG bisects AB perpendicularly. Let that circle be described. Then, the arcs AG, GB being evidently equal, the angles AHG, GHB are equal: and therefore (VI. 3) AH has to HB the same ratio as AD to DB; that is, (const.) the given ratio. The triangle AHB, therefore, is described on the given base AB; has its perpendicular HK equal to DE, and therefore its area equal to the given area; and its sides are in the given ratio.

It is plain, that if CG meet the arc of the semicircle in two points, either of them may be joined with D, and there will be two solutions; if CG touch the semicircle, there will be only one solution; and if it do not meet the circumference, the problem will be impossible.

This method of solution may evidently be employed in reference to several problems, which are usually solved principally by means of proposition G of the sixth book.

Prop. XXV.—To this problem I have given a simple and easy solution, which I believe to be new.

Prop. XXVIII.—From this proposition, and from the sixth and seventh corollaries to the fourth proposition of the fourth book of the Elements, we can derive neat algebraic expressions for the radii of the four circles, each touching the three sides of a triangle. Thus, by dividing the expression for the area by s, we find, according to the sixth corollary, that the radius of the inscribed circle is equal to $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$. In like manner, by dividing the expression for the area successively by $s-a$, $s-b$, $s-c$, we find, according to the seventh corollary, that the radii of the circles touching a, b, c, externally are respectively

$$\sqrt{\frac{s(s-b)(s-c)}{s-a}}, \sqrt{\frac{s(s-a)(s-c)}{s-b}}, \text{ and } \sqrt{\frac{s(s-a)(s-b)}{s-c}}.$$

By taking the continual product of these four expressions, and contracting the result, we get $s(s-a)(s-b)(s-c)$, which is equal to the square of the area: and hence, by expressing this in words, we have the following remarkable theorem: *The continual product of the radii of the four circles, each of which touches the three sides of a triangle, or their prolongations, is equal to the second power of the area.*

PROP. XXIX.—This important problem was proposed by Mr. Townley, and solved by Mr. John Collins, in the Philosophical Transactions for 1671. It is of much use in the surveying of coasts and harbours.

APPENDIX, BOOK IV.

This book is supplementary to the eleventh and twelfth of Euclid, containing a few additional propositions in solid geometry, and exhibiting, in particular, the method of computing the contents of pyramids, cones, and polyhedrons. The integral calculus affords the best and easiest means of investigating most of what remains of the mensuration of the surfaces and volumes of solids.

PROP. IV. COR. 2.—An easy method of computing the content of a truncated pyramid or cone, that is, the frustum which remains when a part is cut from the top by a plane parallel to the base may be thus investigated by the help of algebra. The solid cut off is (APP. IV. 3, schol. 2) similar to the whole; and therefore the areas of their bases will be proportional to the squares of their corresponding dimensions, and consequently to the squares of their altitudes. Hence putting V to denote the volume or content of the frustum, H and B the altitude and base of the whole solid, and h and b those of the solid cut off, if we put qH^2 to denote B, since $B : b :: H^2 : h^2$, or $B : b :: qH^2 : qh^2$, we shall have (V. 14) $b = qh^2$; and therefore (APP. V. 4, cor. 2) the contents of the whole cone and the part cut off are equal respectively to $\frac{1}{3}qH^3$ and $\frac{1}{3}qh^3$; wherefore $V = \frac{1}{3}q(H^3 - h^3)$, or, by resolving the second member into factors, $V = \frac{1}{3}q(H^2 + Hh + h^2)(H - h) = \frac{1}{3}(qH^2 + qHh + qh^2)(H - h)$. Now qH^2 is equal to B, qh^2 to b, qHh to a mean proportional between them, and $H - h$ to the height of the frustum. Hence, *to find the content of a truncated pyramid or cone, add together the areas of its two bases and a mean proportional* *between them, multiply the sum by the height of the frustum, and divide the product by* 3.

This admits of convenient modifications in particular cases. Thus, if the bases be squares of which S and s are sides, and if a be the altitude of the frustum, we shall have

$$V=\tfrac{1}{3}a(S^2+S\,s+s^2)=\tfrac{1}{3}a(3Ss+S^2-2\,Ss+s^2);$$

$$\text{or, } V=\tfrac{1}{3}a\left\{3\,Ss+(S-s)^2\right\}=a\left\{Ss+\tfrac{1}{3}(S-s)^2\right\}.$$

Hence, *to find the content of the frustum of a square pyramid, to the rectangle under the sides of its bases, add a third of the square of their difference, and multiply the sum by the height.* It would be shown in like manner (APP. I. 39, cor. 2) that if R and r be the radii of the bases of the frustum of a cone, and a its altitude,

$$V=a\pi\left\{Rr+\tfrac{1}{3}(R-r)^2\right\}.$$

EXERCISES.

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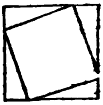
1. The least straight line that can be drawn to another straight line from a point without it, is the perpendicular to it: of others, that which is nearer to the perpendicular is less than one more remote; and only two equal straight lines can be drawn, one on each side of the perpendicular.

2. Of the triangles formed by drawing straight lines from a point within a parallelogram to the several angles, each pair that have opposite sides of the parallelogram as bases, are half of it.

3. If, in proceeding round an equilateral triangle, a square, or any regular polygon, in the same direction, points be taken on the sides, or the sides produced, at equal distances from the several angles, a similar rectilineal figure will be formed by joining each point of section with those on each side of it.



4. If the three sides of one triangle be perpendicular to the three sides of another, each to each, the triangles are equiangular.

5. A trapezoid, that is, a trapezium having two of its sides parallel, is equal to a triangle which has its base equal to the sum of the parallel sides, and its altitude equal to their perpendicular distance.

6. Given the segments into which the line bisecting the vertical angle, divides the base, and the difference of the angles at the base; to construct the triangle, and compute the sides.

7. Within or without a triangle, to draw a straight line parallel to the base, such that it may be equal to the parts of the other sides, or of their continuations, between it and the base.

8. Given the perpendicular of a triangle, the difference of the segments into which it divides the base, and the difference of the angles at the base; to construct the triangle.

9. In the figure for the 47th proposition of the first book of Euclid, prove that CF is perpendicular to AD.

10. The angle made by two chords of a circle, or by their continuations, is equal to an angle at the circumference standing on an arc equal to the sum of the arcs intercepted between the chords, if the point of intersection be within the circle, or to their difference, if it be without: (2) the angle made by a tangent and a line

cutting the circle, is equal to an angle at the circumference on an arc equal to the difference of the arcs intercepted between the point of contact and the other line: and (3) the angle made by two tangents is equal to an angle at the circumference standing on an arc equal to the difference of those into which the circumference is divided at the points of contact.

Cor. If a tangent be parallel to a chord, the arcs between the point of contact, and the extremities of the chord are equal.

11. Given the sum of the perimeter and diagonal of a square; to construct it.

12. On a given straight line describe a square, and on the side opposite to the given line, describe equilateral triangles lying in opposite directions; circles described through the extremities of the given line, and through the vertices of these triangles are equal.

13. To inscribe an equilateral triangle in a given square.

14. Given the angles and two opposite sides of a trapezium to construct it.

15. In a given circle to place two chords of given lengths, and inclined at a given angle.

16. In the second figure in page 83, if AM, BM be joined, the angle AMB is half of ACB.

17. If, in proceeding in the same direction round any triangle, as in Exercise 3, points be taken at distances from the several angles, each equal to a third of the side, the triangle formed by joining the points of section is one third of the entire triangle.

18. To describe a square having two of its angular points on the circumference of a given circle, and the other two on two given straight lines drawn through the centre. Show that there may be eight such squares.

19. Given the vertical angle of a triangle, and the segments into which the base is divided at the point of contact of the inscribed circle; to describe the triangle, and compute the sides.

20. If any three angles of an equilateral pentagon be equal, all its angles are equal.

21. Given two sides of a triangle, and the difference of the segments into which the third side is divided by the perpendicular from the opposite angle; to construct the triangle.

22. Given the vertical angle of a triangle, the line bisecting it, and the perpendicular; to construct the triangle.

23. From a given centre to describe a circle, from which a straight line, given in position, will cut off a segment containing an angle equal to a given angle.

24. In a given triangle to inscribe a semicircle having its centre in one of the sides.

25. Through three given points to draw three parallels, two of which may be equally distant from the one between them.

26. Given an angle of a triangle, and the radii of the circles touching the sides of the triangles into which the straight line bisecting the given angle divides the triangle; to construct it.

27. In a rhombus to inscribe a square.

28. Given the lengths of the two parallel sides of a trapezoid, and the lengths of the other sides; to construct it.

29. Given one of the angles at the base, and the segments into which the base is divided at the point of contact of the inscribed circle; to describe the triangles.

30. To draw a tangent to a given circle, such that the part of it intercepted between the continuations of two given diameters may be equal to a given straight line.

31. To draw a tangent to a given circle, such that the part of it between two given tangents to the circle may be equal to a given straight line.

32. Given the vertical angle of a triangle, the difference of the sides, and the difference of the segments into which the line bisecting the vertical angle divides the base; to construct the triangle.

33. Given any three of the circles mentioned in the scholium to the fourth proposition of the fourth book of Euclid; to describe the triangle.

34. A straight line and a point being given in position, it is required to draw through the point two straight lines inclined at a given angle, and enclosing with the given line a space of given magnitude.

35. From two given straight lines to cut off equal parts, each of which will be a mean proportional between the remainders.

36. A square is to a regular octagon described on one of its sides, as 1 to $2(1+\sqrt{2})$.

37. In a given triangle to inscribe a parallelogram of a given area.

38. In a given circle to inscribe a parallelogram of a given area.

39. Through a given point between the lines forming a given angle, to draw a straight line cutting off the least possible triangle.

40. To divide a given straight line into two parts, such that the square of one of them may be double of the square of the other, or may be in any given ratio to it.

41. To produce a given straight line, so that the square of the whole line thus produced, may be double of the square of the part added, or in any given ratio to it.

42. Given the area of a right-angled triangle, and the sum of the legs; to construct it.

43. Given the area and the difference of the legs of a right-angled triangle; to construct it.

44. Given one leg of a right-angled triangle, and the remote segment of the hypotenuse, made by a perpendicular from the right angle; to construct the triangle.

45. Given the base of a triangle, the vertical angle, and the side of the inscribed square standing on the base; to describe the triangle.

46. Given the base of a triangle, and the radii of the inscribed and circumscribed circles; to construct the triangle.

47. To draw a chord in a circle which will be equal to one of the segments of the diameter that bisects it.

48. In a given circle to draw a chord which will be equal to the

difference of the parts into which it divides the diameter that bisects it.

49. If two sides of a regular octagon, between which two others lie, be produced to meet, each of the produced parts is equal to a side of the octagon, together with the diagonal of a square, described on the side.

50. The perimeter of a triangle is to the base, as the perpendicular to the radius of the inscribed circle.

51. From a given point without a given circle, to draw a straight line cutting the circle, so that the external and internal parts may be in a given ratio.

52. Each of the complements of the parallelograms, about the diagonal of a parallelogram, is a mean proportional between those parallelograms.

53. Given the ratio of two straight lines, and the difference of their squares; to find them.

54. The square of the perimeter of a right-angled triangle is equal to twice the rectangle under the sum of the hypotenuse and one leg. and the sum of the hypotenuse and the other.

55. The quadrilateral formed by straight lines bisecting each pair of adjacent sides of a quadrilateral, is a parallelogram which is half of the quadrilateral; and straight lines, joining the points in which the sides of that parallelogram are cut by the diagonals of the primitive figure, form a quadrilateral similar to that figure and equal to a fourth of it.

56. From three given points as centres, and not in the same straight line, to describe three circles each touching the other two. Show that this admits of four solutions.

57. To add a parallelogram to a rhombus, such that the whole figure may be a parallelogram similar to the one added.

58. A straight line being given in position, and a circle in magnitude and position, it is required to describe two equal circles touching one another, and each touching the straight line and the circle.

59. The squares of the diagonals of a quadrilateral are together double of the squares of the straight lines joining the points of bisection of the opposite sides.

60. In a given rhombus to inscribe a rectangle having its sides in a given ratio.

61. If, through the vertex and the extremities of the base of a triangle, two circles be described intersecting one another in the base or its continuation, their diameters are proportional to the sides of the triangle.

62. To draw a straight line cutting two given concentric circles, so that the parts of it within them may be in a given ratio.

63. From a given point, within a given circle, or without it, to draw two straight lines to the circumference, perpendicular to one another, and in a given ratio. When will this be impossible?

64. If a straight line be divided in extreme and mean ratio, the squares of the whole and the less part are together equal to three times the square of the greater.

65. If a straight line be cut in extreme and mean ratio, and be

also bisected, the square of the intermediate part, and three times the square of half the line are equal to twice the square of the greater part.

66. If the hypotenuse of a right-angled triangle be given, the side of the greatest inscribed square, standing on the hypotenuse, is one third of the hypotenuse.

67. To divide a given semicircle into two parts by a perpendicular to the diameter, so that the radii of the circles inscribed in them, may be in a given ratio.

68. To draw a straight line parallel to the base of a triangle, making a segment of one side equal to the remote segment of the other.

69. In the figure for the 11th proposition of the second book of Euclid, the square of the diameter of the circle, passing through the points F, H, D, is six times the square of the straight line joining FD.

70. Given the base, the area, and the sum of the squares of the sides of a triangle; to construct it.

71. If, from the extremities of the hypotenuse of a right-angled triangle as centres, arcs be described passing through the right angle, the hypotenuse is divided into three segments, such that the square of the middle one is equal to twice the rectangle of the others.*

72. On a given hypotenuse to describe a right-angled triangle, such that the difference between one leg and the adjacent segment of the hypotenuse made by a perpendicular from the right angle, may be a maximum.

73. On a given hypotenuse to construct a right-angled triangle, such that one segment of the hypotenuse made by the perpendicular, from the right angle, may be equal to the sum of the perpendicular, and the other segment.

74. The square of CD (see the figure for proposition XXV, page 248) is equal to the rectangle MK.NL; and if the circles touch one another externally, CD is a mean proportional between their diameters. Also the square of FG is equal to the rectangle ML.NK.

75. On a given straight line to describe an isosceles triangle, having the vertical angle treble of each of the angles at the base.

76. If BD, CE, in the figure in page 245, be joined, the interior figure contained by the segments of the diagonals, is a regular pentagon; and if each pair of remote sides AB, DC, &c. be produced to meet, their points of intersection will be the angular points of another regular pentagon.

77. To find a point from which, if straight lines be drawn to three given points, they will be proportional to three given straight lines.

78. Given the segments into which the base of a triangle is divided by two straight lines trisecting the vertical angle; to construct it.

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* Compare this with Appendix, Book I. Prop. XXVII. Schol.

79. If one diagonal of a quadrilateral inscribed in a circle, be bisected by the other, the square of the latter is equal to half the sum of the squares of the four sides.

80. To divide a straight line into two parts, such that the squares of the whole and one of the parts may be double of the square of the other part.

81. Given the segments into which the base of a triangle is divided by the straight line bisecting the vertical angle, to construct the triangle so that its angle adjacent to the greater segment may be either of a given magnitude, or a maximum, and in each case to compute the remaining sides and angles.

82. To draw a straight line bisecting a given parallelogram, so that if it be produced to meet the sides produced, the external triangles will have a given ratio to the parallelogram.

83. Through a given point to draw a straight line, which, if continued, would pass through the point of intersection of two given inclined straight lines, without producing those lines to meet.

84. If, from any point in the circumference of the circle described about an equilateral triangle, chords be drawn to its three angles, the sum of their squares is equal to six times the square of the radius of the same circle, or to twice the square of a side of the triangle.

85. Given the difference of the angles at the base of a triangle, the difference of the segments into which the base is divided by the perpendicular, and the ratio of the sides; to construct the triangle.

86. Given the base and vertical angle of a triangle, to construct it, so that the line bisecting the vertical angle may be a mean proportional between the segments into which it divides the base.

87. Given two sides of a triangle, and the ratio of the base and the line bisecting the vertical angle; to construct the triangle.

88. If the vertical angle of a triangle be double of one of the angles at the base, the rectangle under the sides is equal to the rectangle under the base, and the line bisecting the vertical angle.

89. If a straight line be divided into parts, which, taken in succession, are continual proportionals, and if circles be described on the several parts as diameters, a straight line which touches two of the circles on the same side of the straight line joining their centres, will touch all the others.

90. Given the segments into which the base is divided by the straight line bisecting the vertical angle, and the angles which that straight line makes with the base; to construct the triangle.

91. To divide a given circle into two segments, such that the squares inscribed in them may be in a given ratio.

92. Through a given point in the base of a given isosceles triangle, or its continuation, to draw a straight line such that the lines intercepted on the equal sides, or their continuations between that line and the extremities of the base, may have one of the equal sides as a mean proportional between them.

93. Through a point in the circumference of a given circle, to draw two chords, such that their rectangle may be equal to a given

space, and the chord joining their other extremities equal to a given straight line.

94. In any triangle the radius of the circumscribed circle is to the radius of the circle which is the locus of the vertex, when the base and the ratio of the sides are given, as the difference of the squares of those sides is to four times the area.

95. The difference of the sides of a triangle is a mean proportional between the difference of the segments into which the base is divided by the perpendicular, and the difference of those into which it is divided by the line bisecting the vertical angle.

96. Let the angles of a parallelogram which has unequal sides, be bisected by straight lines cutting the diagonals, and let the points of intersection be joined: the figure thus formed is a parallelogram, which has to the proposed parallelogram the duplicate ratio of that which the difference of the unequal sides of the latter has to their sum.

97. A circle and a point being given, it is required to describe a triangle similar to a given one, having its vertex at the given point, and its base a chord of the given circle.

98. Given the three points in which the sides of a triangle are cut by the perpendiculars from the opposite angles; to construct the triangle.

99. Given the angles of a triangle, and the lengths of three straight lines drawn from the angular points to meet in another point; to construct the triangle.

100. Given the base of a triangle, and the ratio of its sides; to construct it, so that the distance of its vertex from a given point may be a maximum or minimum.

101. To divide a circle into two segments, such that the sum of the squares inscribed in them may be equal to a given space.

102. Through a given point, with a given radius, to describe a circle bisecting the circumference of a given circle.

103. With a given radius to describe a circle bisecting the circumferences of two given circles.

104. In a right-angled triangle, the rectangle under the radius of the inscribed circle, and the radius of the circle touching the hypotenuse and the legs produced, is equal to the area. So, likewise, is the rectangle under the circles touching the legs externally, and the continuations of the other sides.

105. If three straight lines be continual proportionals, the sum of the extremes, their difference, and double the mean, will be the hypotenuse and legs of a right-angled triangle.

106. From two given centres, to describe circles having their radii in a given ratio, and the part of their common tangent, between the points of contact, equal to a given straight line.

107. In a given circle to inscribe a quadrilateral, having two of its sides equal to two given lines, and the other two in a given ratio.

108. With a given radius to describe a circle touching two given circles. When is this impossible, and when are there two, or more solutions?

109. Through a given point to describe a circle touching a given circle, and having its centre in a given straight line. When is this impossible?

110. With a given radius, to describe a circle passing through a given point, and touching a given straight line.

111. To trisect a given triangle by straight lines drawn from a given point within the triangle, one of those lines being given.

112. Given the angles and the vertex of a triangle, to describe it, so that the extremities of the base may be on two straight lines given in position.

113. To divide a given chord of a given circle into two parts, such that their rectangle may have a given ratio to the square of the straight line drawn from the point of section to a given point in the circumference.

114. To describe two circles touching each other, and each of them touching the base and one of the remaining sides of a given triangle, the ratio of their radii being given.

115. Two straight lines being given in position, and two points being given in one of them; it is required to find two points in the other, such that if two straight lines be drawn from the given points to the latter, and two from the latter to meet in the other line, the sum of all the four will be a minimum.

116. Let BEF (see the figure for proposition G of the sixth book) be a given triangle, and (VI. G, cor.) through B let the circle ABC be described, such that straight lines drawn from any point in its circumference may be proportional to BE, BF: then, if through E and F two other circles be described, the first, such that straight lines drawn from B and F, to meet in its circumference, may be proportional to EB, EF, and the second, such that straight lines drawn from E and B, to meet in its circumference, may be proportional to FE, FB; the centres of the three circles will lie in the same straight line, and there are two points through which all their circumferences pass.

117. In the figure for proposition H of the sixth book, BF : FC :: AC.BD : AB.DC.

118. To describe a triangle having its sides equal to three given straight lines, and passing through three given points.

119. On a given straight line as hypotenuse, to describe a right angled triangle, such that, of the triangles into which it is divided by the perpendicular from the right angle, one may be a maximum.

120. In a given triangle ABC, (see the first figure for the second proposition of the sixth book) draw DE parallel to BC, so that BE being joined, the triangle BDE may be a maximum.

121. To describe a circle having its centre in a given line, and bisecting the circumferences of two given circles.

122. A straight line and two points equally distant from it, on the same side, being given in position, it is required to draw through the points two straight lines forming with the given line the least isosceles triangle possible, on the side on which the points are.

123. To describe a circle touching a diameter of a given circle

in a given point, and having its circumference bisected by that of the given one.

124. If an angle of a triangle be 60°, the square of the opposite side is less than the squares of the other two by their rectangle: but if an angle be 120°, the square of the opposite side is greater than the squares of the others by their rectangle.

125. In the figure in page 249, prove that the three straight lines joining AF, BE, CD are all equal.

126. The chord of 120° is equal to the tangent of 60°.

127. The sines of the parts into which the vertical angle of a triangle is divided by the straight line bisecting the base, are reciprocally proportional to the adjacent sides. Show from this how a given angle may be divided into two parts, having their sines in a given ratio.

128. The diameter of the circle described about any triangle is equal to the product of any side and the cosecant of the opposite angle.

129. In any triangle ABC, the radius of the inscribed circle is equal to $a.\frac{\sin\frac{1}{2}B \sin\frac{1}{2}C}{\cos\frac{1}{2}A}$, to $b.\frac{\sin\frac{1}{2}A \sin\frac{1}{2}C}{\cos\frac{1}{2}B}$, or to $c.\frac{\sin\frac{1}{2}A \sin\frac{1}{2}B}{\cos\frac{1}{2}C}$; or, finally, to the cube root of

$$abc \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C.$$

130. Given the sum of the tangents, and the ratio of the secants, of two angles to a given radius; to determine the angles geometrically, and by computation.

131. To divide a given angle into two parts having their tangents in a given ratio.

132. Find an angle, such that its tangent is to the tangent of its double, in a given ratio; suppose that of 2 to 5.

133. The square of the diameter of a globe is three times the square of the side of the inscribed cube.

SKETCH OF THE HISTORY

OF

ELEMENTARY GEOMETRY.

WITH a few of the facts and relations of Geometry, men must have been acquainted from the earliest period. It is impossible to ascertain, however, in what country the elements of the science first began to be reduced to anything like a systematic form; but, while there is reason to believe, that the ancient Hindoos and Egyptians did something in this way, we have no authentic record regarding anything that was done on the subject, prior to the time at which the study was introduced in ancient Greece, upwards of six hundred years before the commencement of the Christian era. In that country, and in its dependencies, polite literature and the fine arts were cultivated at a very early period with extraordinary zeal and success; and, at the same time, the ingenious and intellectual people of the same regions made great progress in enriching and extending geometry, almost the sole branch of pure science to which the energies of the human mind were directed at an early period in the history of man.

Thales of Miletus, in Asia Minor, who is said to have been born about 640 years before Christ, is generally regarded as the first who introduced the study of geometry as a branch of science among the Greeks; and it is probable, that he got his first notions of the subject in Egypt, in which country he is said to have resided for some time, and to have formed an acquaintance with its priests, the sole depositories of whatever science existed in the land. This philosopher, having returned to his native country, Ionia, founded in it what has thence been called the Ionian school.

The celebrated Pythagoras, a native of the island of Samos, is said to have been born about 580 years before Christ. This distinguished individual, after having been a pupil of Thales, increased his knowledge by travelling in Egypt and India, and making himself acquainted with the learning which then existed in these countries. To him has been attributed the discovery of the two important propositions, the 32d, and the 47th, of the first book of Euclid. Some have thought, however, that he learned them on his travels.*

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* Even if Pythagoras did discover these propositions, it is highly improbable, that, as has been related, he sacrificed, in his joy on discovering the 47th, a hundred

Another mathematician of considerable celebrity was Hippocrates, of the island of Chios, who lived above 400 years before Christ. He is the first who is known to have succeeded in effecting the quadrature of a curvilineal space, by finding a rectilineal one equal to it. This ingenious, though barren discovery, is given in the 43d proposition of the first book of the Appendix.

One of the most distinguished promoters of science among the Greeks was the celebrated philosopher Plato. There is no decisive evidence, that he made mathematical discoveries himself; though some have supposed, that geometrical analysis (see Appendix, book II. prop. 10, &c.) as a separate subject of study, and the conic sections, which were certainly first studied in his school, were indebted to him for their origin. However this may be, he greatly encouraged the study of geometry among his pupils, all of whom he required to have a competent knowledge of the science, and he even put an inscription over the door of his school, stating, that no one unacquainted with geometry should be allowed to enter it as a pupil. The conic sections, as the name, in part at least, implies, are those curves which are produced by the section of a cone by a plane: and the discovery of these important lines is said to have originated in the attempts that were made to solve the celebrated problem, of *finding two mean proportionals between two given magnitudes.**

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oxen to the muses. So expensive a sacrifice, there is good reason to believe, would have been inconsistent with his worldly circumstances: and the shedding of blood was at variance with his principles; as, in consequence of his holding the doctrine of the transmigration of souls, he considered it inhuman to kill creatures which might possibly be animated by the souls of the fathers, brothers, or other relations of the persons who sacrificed them. Thus, Ovid represents him, in the fifteenth book of the Metamorphoses, as saying,

> Corpora, quæ possint animas habuisse parentum,
> Aut fratrum, aut aliquo junctorum fœdere nobis,
> Aut hominum certè, tuta esse et honesta sinamus.
>
> * * * * *
>
> Bos aret; aut mortem senioribus imputet annis;
> Horriferum contra Borean ovis arma ministret;
> Ubera dent saturæ manibus pressanda capellæ, &c.

Pythagoras paid much attention also to astronomy, music, and arithmetic; and he was certainly one of the most distinguished men of ancient times.

* The method of finding *one* mean proportional between two straight lines is given in the 13th proposition of the sixth book of Euclid. The finding of *two* mean proportionals, however, by means of elementary geometry, that is, by employing only the straight line and the circle, long exercised the ingenuity of geometricians, but in vain; and all the well-informed mathematicians of modern times agree, that the solution cannot be effected without the aid, directly or indirectly, of the higher geometry. The solution is easily effected by means of the conic sections and many other curves. It is easily obtained, also, by means of algebra. Thus, let a and b be the given magnitudes, and 1 to x the ratio of each of the proportionals to the one following it. Then, as $1 : x :: a : ax$, the first mean; as, $1 : x :: ax : ax^2$, the second; and $1 : x :: ax^2 : ax^3$; which, by the nature of this problem, is equal to b. Putting it, therefore, equal to b, dividing by a, and extracting the cube root, we find the value of x; and thence the values of the means, ax and ax^2, are obtained at once. If, for example, it were required to find two mean proportionals between two lines of 54 and 16 inches respectively, we get by dividing the latter by the former, $\frac{8}{27}$, the cube root of which is $\frac{2}{3}$. Hence, the first mean is 36, two thirds of 54; and the second 24, two thirds of 36.

Another celebrated problem which occupied much attention in the school of Plato, and in subsequent times, was *the trisection of an angle.* The geometricians of that school failed, as all others have done, in solving this problem by means of elementary geometry. While they failed, however, in their main object, both in reference to this problem, and that of finding two mean proportionals, their time and exertions were not thrown away, as they made valuable discoveries regarding the conic sections, as already mentioned, and also in other branches of geometry.*

About 333 years before Christ, and 15 years after the death of Plato, the city of Alexandria in Egypt was founded by Alexander the Great. This city was soon after made the royal residence by Ptolemy Lagus, called also Ptolemy Soter, king of Egypt, and that prince established its celebrated Museum. This was an academy in which a society of learned men devoted themselves to the study of science; and it continued to be the asylum of learned men, and the chief seat of mathematics, philosophy, and literature in general, for nearly 1000 years, till the city was taken by the Saracens in the 640th year of the Christian era.

Of the distinguished men who were connected with this establishment, the one who claims most attention in this sketch, is Euclid, the author of the Elements, and several other works. Where this mathematician was born, and the times of his birth and death are all unknown. It is believed, however, that he was born upwards of 300 years before the commencement of our era,

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According to an ancient tradition, the problem of finding two mean proportionals had its origin in a response given by the oracle of Apollo at Delos (whence it was often called the *Delian Problem*) in reply to an inquiry regarding the means of averting a plague, by which Attica was desolated, in consequence of an affront alleged to have been offered by the Athenians to that deity. In the response, they were ordered to *double the altar;* and, as the altar of Apollo at Athens was of a cubical form, they were thus required to *double a cube.* It was soon seen, that if they doubled the length, breadth, and height of the altar, its magnitude would be increased eight fold; and it was discovered, it is said, by Hippocrates of Chios, that if each of these dimensions were increased in the ratio of 1 to the first of two mean proportionals between 1 and 2, the content or volume of the altar would be doubled.

* The mode of bisecting an angle is given by Euclid in the ninth proposition of the first book; and that of trisecting a right angle, or any angle obtained by bisecting a right angle, its half, its fourth part, &c. is pointed out in the seventh corollary to the thirty-second proposition of the same book, and in the note to that corollary. The trisection of other angles has been often attempted by means of elementary geometry, but in vain. By means, however, of the conic sections, as well as of many other curves, the trisection may be readily effected; and the same may be easily done by means of algebra in connexion with trigonometrical principles. To show how the solution may be obtained in the last-mentioned way, let 3A be changed into θ in the second of each of the sets of formulas marked (235) and (239) in my Elements of Plane and Spherical Trigonometry; and there will be obtained,

$$4\cos^3\tfrac{1}{3}\theta - 3\cos\tfrac{1}{3}\theta - \cos\theta = 0, \text{ and } 4\sin^3\tfrac{1}{3}\theta - 3\sin\tfrac{1}{3}\theta + \sin\theta = 0.$$

Then the resolution of the first of these equations, which is a cubic in reference to $\cos\frac{1}{3}\theta$, or of the second, which is also a cubic in reference to $\sin\frac{1}{3}\theta$, will determine $\frac{1}{3}\theta$ by means of its cosine in the one case, and its sine in the other, when θ is given. Each equation will have three roots, which in the first will be the cosines, and in the second the sines, of one third of θ, one third of $\theta+360°$, and one third of $\theta+720°$. Other solutions might be obtained from formula (47) or (48) of the same work.

and he was the preceptor and friend of Ptolemy Philadelphus, king of Egypt.* Of the works which he composed, the most celebrated is the Elements already referred to; a work which has been translated into the languages of all nations that have made any considerable progress in civilization, since it was first published, and which has been more generally used for the purposes of teaching, than any work on abstract science that has ever appeared.

What share Euclid had in the discovery of the propositions contained in this work, it is impossible to determine. It is known, that before his time, Hippocrates of Chios, Eudoxus, and others, had composed treatises on elementary geometry, which were highly esteemed, and which must doubtless have contained most of the principal propositions which we find in Euclid. Besides this, the progress made in the theory of the conic sections before his time proves incontrovertibly, that much must have been done in elementary geometry, as its leading propositions are required in establishing that theory. With regard to the composition of the work, however, Euclid is entitled to the greatest credit. His reasoning is conducted with extreme rigour, and the work exhibits throughout such accuracy of thought and expression as is calculated to produce the most beneficial effects on the mind of the careful reader. It is a decisive proof, also, of the estimation in which it has been held, that it has been handed down to modern times, while the similar works that preceded it have been lost; and that even amid the gigantic strides made in modern science, no work has yet appeared which has superseded it in any considerable degree.

This celebrated work is divided into fifteen books. Of these, the first four treat of the equality or inequality of straight lines, of angles, and of figures bounded by straight lines or circular arcs. The fifth contains the theory of proportion in the abstract, and the sixth applies this theory in reference to the magnitudes treated of in the first four books. The seventh, eighth, and ninth books treat of the properties of numbers, and form the most ancient tract on arithmetic extant. In the present state of science, however, these books are of scarcely any value. The tenth is devoted to the consideration of commensurable and incommensurable magnitudes, and the last five treat of solids. The last two of these are supposed to have been written, not by Euclid, but by Hypsicles of Alexandria, who lived about 200 years after him. The first six books, and the eleventh and twelfth, form the most valuable part of the work—the part which is found to be of importance in science in its present state; and accordingly most of the modern editions contain only these. The editions of Barrow, Gregory, and Peyrard, besides others, contain the whole fifteen books.

Since the time of Euclid many curious and interesting propositions in elementary geometry have been discovered by different individuals. Many of the most valuable of these will be found in the Appendix to this work, and in the propositions in the second and sixth books, marked A, B, C, &c. In reference to these matters, it is unnecessary to go into minute details; and it may be sufficient to state, that the most interesting addition to elementary geometry, made in modern times (too difficult, however, in its character to be suited to the Appendix to this work), was given by Professor Gauss of Göttingen, in his *Disquisitiones Arithmeticæ*, published in 1801. The only regular polygons that could be inscribed in a given circle before the time of Gauss, were those which are treated of in the fourth book of Euclid, and those which are obtained by continued bisections of the arcs of which their sides are chords. That distinguished mathematician, however, discovered the means of inscribing polygons of 17 sides, of 257 sides; and, in general, of 2^n+1 sides, when 2^n+1 is a prime number.

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* A remarkable reply which he is said to have given to Ptolemy, is recorded, and is often referred to. That prince having asked Euclid whether there was no shorter or easier means of acquiring a knowledge of geometry, than that which he had given in his Elements, Euclid is reported to have said in reply,—"No, Sire, there is no royal road to geometry."

Several treatises on Trigonometry were written in ancient times. The earliest of these of which we have any record, was by Hipparchus, the celebrated astronomer, who flourished about 150 years before Christ. This has been lost, however; and the earliest treatise extant is that of Theodosius, who lived soon after Hipparchus. This treatise, as well as some others of ancient times, is confined to spherical trigonometry, that branch of the subject which is chiefly useful in astronomy; and the demonstrations are conducted in the same synthetic manner that is employed by Euclid in the Elements. Other writers on trigonometry were Menelaus, who lived in the first century of the Christian era; and Ptolemy, the Egyptian astronomer, from whom the Ptolemaic system takes its name, and who gave some useful matter regarding trigonometry in the first book of his Almagest (Μεγαλη Συνταξις). In trigonometrical computations the ancients used chords instead of sines; the use of the latter, which is much preferable, having been introduced by the Arabians in the middle ages.

Great improvements were introduced, about the middle of the fifteenth century, by Purbach, and afterwards by his pupil, John Muller, commonly called Regiomontanus, both Germans. After these writers, many others, particularly Vieta, contributed in succession to the gradual improvement of trigonometry. It was reserved, however, for Baron Napier, or Neper, of Merchiston, near Edinburgh, to confer upon science in general, and on trigonometry in particular, vast advantages, by his invention of logarithms, in the year 1614. The use of these remarkable numbers changed, in a great degree, the whole structure of trigonometry; and, besides the important aid which they afford, in many instances in scientific investigations, they greatly facilitate computations in general, but more particularly in trigonometry, navigation, and astronomy.

During the period that has elapsed between the invention of logarithms and the present time,*—the most brilliant period, by far, in the history of science,—the period in which the discoveries of Newton, Euler, La Grange, Laplace, and many other men of the highest genius, have been given to the world—during this period — the greatest stride that has been made in the advancement of trigonometry and of geometry at large, has been the bringing of them both under the dominion of algebra. By this means, the investigations are freed from the cumbrous formality and tediousness of demonstrations such as those of Euclid, which, however admirable they are in the elements of science, and however well they are calculated to strengthen and discipline the mind, are powerless in the more advanced researches of science. By these means, trigonometry has been enriched with numberless new additions; and, instead of being confined, as it formerly was, to the mere resolution of triangles, it has now become a powerful instrument of investigation in other departments of science. Algebra, too, applied in the way first pointed out by Descartes, and followed up by the powerful means afforded by the differential and integral calculus, has given the higher geometry an extent incomparably greater than it could possibly have attained by the ancient geometrical method of investigation. Within a very recent period, this application of algebra has been reduced to a systematic form, constituting a new, or at least a separate, branch of science, which is generally called *Analytic Geometry.*

For farther information on these curious and important subjects, the student may have recourse to Montucla's and Bossut's Histories of Mathematics, to Hutton's and Barlow's Mathematical Dictionaries, to the Transactions of scientific societies, to encyclopœdias, and to various biographical works.

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* It may be worth mentioning, that, during this period, some methods of resolving the elementary cases of trigonometry have been introduced, which are of value in a practical point of view. Thus, the very elegant formulas in the corollary to the 6th proposition in page 166, and those in the first and second corollaries in the next page, were discovered by William Purser of Dublin, about the year 1632, or soon after. The properties established in the 4th proposition, page 165, were given for the first time, so far as I have been able to ascertain, in Thacker's Miscellany in 1643; and their use in resolving the second case in plane trigonometry was pointed out by the late Professor Wallace of Edinburgh, in the Transactions of the Royal Society of Edinburgh for 1823. The method of resolving the third case, given in page 171, as arising from the 2d and 3d corollaries to the 7th proposition, page 167, was first given in the second edition of my Elements of Plane and Spherical Trigonometry, published in 1830: and in the next edition of this work a new formula (83) was given, which answers the same purpose with equal facility.

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